ON ESTIMATING THE NUMBER OF SOLUTIONS OF CERTAIN EQUATIONS

MATHEMATICS

1965

SovietRxiv

Abstract

Full Text

UDC 511.9

MATHEMATICS

A. A. KARATSUBA

ON ESTIMATING THE NUMBER OF SOLUTIONS OF CERTAIN EQUATIONS

(Presented by Academician Yu. V. Linnik on March 23, 1965)

Consider the equation

$$x_1^n + \dots + x_k^n = y_1^n + \dots + y_k^n, \tag{1}$$

where $P > \exp n^6,\ 1 \leq x_i, y_i \leq P,\ i = 1, 2, \dots, k$. Let $I_{k,n}(P)$ be the number of solutions of this equation. An upper estimate for $I_{k,n}(P)$ is closely connected with the asymptotic formula in Waring' s problem (see, for example, ([4]), p. 111). By the method he created, I. M. Vinogradov obtained, for $I_{k,n}(P)$ with $k \geq 4n^2 \ln n$, the estimate

$$I_{k,n}(P) \leq C(k,n) P^{2k-n}, \tag{2}$$

where $C(k,n)$ is a constant depending only on n and k.

Let us note that

$$I_{k,n}(P) \geq (2k)^{-1} P^{2k-n}. \tag{3}$$

Trivially we have

$$I_{k,n}(P) \leq P^{2k-1}.$$

It follows from the Hardy–Littlewood theorem that for $k \geq 2$ one has

$$I_{k,n}(P) \leq P^{2k-2+\varepsilon}.$$

The known problem of an asymptotic formula in Waring' s problem reduces to obtaining the estimate (2) for $k \geq 4n + 1$ or $k \geq n^{1+\varepsilon}$ (see ([2],[4])). In this article the question of estimates of type (2) is studied for possibly smaller k.

In what follows we shall use the following notation: $k, m, n, r, s, t, l, x, y, \lambda, P$ are integers; $n \geq 2$; $2 \leq r \leq n$; $I_{k,n}(P)$ is the number of solutions of equation (1); $c, c_1, c_2, \ldots$ are constants that may depend on n and k; $A, A_1, A_2, \ldots$ are absolute constants.

Theorem 1. For $k \geq 6rn\ln n$, the inequality

$$I_{k,n}(P) \leq c_1 P^{2k-r} \tag{4}$$

holds.

Theorem 2. Let $1 \leq m \leq n$. Then, for $k \geq 6mn\ln n$, we have

$$I_{k,n}(P) \leq c_2 I_{k,m}(P). \tag{5}$$

Theorems 1 and 2 are easily proved from the following main theorem.

Main theorem. Let $p > \exp n^6$ be a prime number and let p_i be prime numbers satisfying the inequalities:

$$\frac{1}{4}p_i^{1-1/r} < p_{i+1} < \frac{1}{2}p_i^{1-1/r}, \qquad i = 1, 2, \ldots, \tau = [5r\ln n]; \qquad p_1 = p.$$

Let, further, $q_1 = p_1^r \ldots p_\tau^r$ and $c_3 q_1^{1/r} \leq P \leq c_4 q_1^{1/r}$. Consider the system of congruences

$$\left.\begin{array}{c} x_1 + \ldots - y_k \equiv \lambda_1 \\ \vdots \\ x_1^n + \ldots - y_k^n \equiv \lambda_n \end{array}\right\} \pmod{q_1},$$

$$1 \leq x_i,\ y_i \leq P, \qquad i = 1, 2, \ldots, k.$$

If $N_k^{(q_1)}(\lambda_1, \ldots, \lambda_n)$ is the number of solutions of this system of congruences, then for $k \geqslant 6rn\ln n$ the estimate holds

$$N_k^{(q_1)}(\lambda_1, \ldots, \lambda_n) \leqslant N_k^{(q_1)}(0, \ldots, 0) \leqslant c_5 P^{2k-rn+r(r-1)/2}.$$

Proof of Theorem 1. Consider the congruence

$$x_1^n + \ldots + y_k^n \equiv 0 \pmod{q}, \qquad 1 \leqslant x_i,\ y_i \leqslant P, \quad i = 1, 2, \ldots, k,$$

and let $N_{kn}^{(q)}(P)$ be the number of solutions of this congruence. Obviously,

$$I_{k,n}(P) \leqslant N_{k,n}^{(q)}(P).$$

Further we have

$$N_{k,n}^{(q)}(P) = \sum_{\lambda_1,\dots,\lambda_{n-1}} N_{k,n}^{(q)}(\lambda_1,\dots,\lambda_{n-1},0) \leqslant N_{k,n}^{(q)}(0,\dots,0) \sum_{\lambda_1,\dots,\lambda_{n-1}} 1.$$

Putting $q = q_1$ and applying the main theorem, we obtain the required result.

Proof of Theorem 2. Using inequality (4), for $k \geqslant 6mn \ln n$ we have

$$I_{k,n}(P) \leqslant c_6 P^{2k-m}.$$

For $I_{k,m}(P)$ it is easy to obtain a lower estimate, for any k, of the form

$$I_{k,m}(P) \geqslant c_7 P^{2k-m}.$$

The assertion of the theorem follows from these two inequalities.

Inequalities of the type (4) and (5) also hold for systems of equations.

Theorem 3. Consider the system of equations

$$\begin{gathered} x_1^{m_1} + \dots - y_k^{m_1} = 0, \\ \dots\dots\dots\dots\dots \\ x_1^{m_s} + \dots - y_k^{m_s} = 0, \\ x_1^n + \dots - y_k^n = 0, \\ 1 \leqslant x_i,\ y_i \leqslant P, \qquad i = 1,2,\dots,k, \\ 1 \leqslant m_1 < m_2 < \dots < m_s < m_{s+1} = n, \qquad 1 \leqslant s \leqslant n. \end{gathered} \tag{6}$$

Let r be an arbitrary integer, $1 \leqslant r \leqslant n$, and let the integer t be determined by the inequalities

$$m_t \leqslant r < m_{t+1}.$$

Then, for $k \geqslant 6rn \ln n$, for the number of solutions I_k' of this system the estimate

$$I_k' \leqslant cP^{2k-\delta}, \qquad \text{where } \delta = m_1 + \dots + m_t + (s-t+1)r,$$

holds.

Theorem 4. Along with the system (6), consider the system of equations:

$$\begin{gathered} x_1^{n_1} + \dots - y_k^{n_1} = 0, \\ \dots\dots\dots\dots\dots \\ x_1^{n_l} + \dots - y_k^{n_l} = 0, \\ 1 \leqslant x_i,\ y_i \leqslant P, \qquad i = 1,2,\dots,k, \\ 1 \leqslant n_1 < n_2 < \dots < n_l \leqslant n, \end{gathered}$$

and let I_k'' be the number of solutions of this system. Define the smallest r by the inequalities

$$m_t \leqslant r < m_{t+1}, \qquad n_1 + \dots + n_l \leqslant m_1 + \dots + m_t + r(s-t+1).$$

Then for $k \geqslant 6rn \ln n$ the relation

$$I'_k \leqslant c_8 I''_k$$

holds.

Theorems 3 and 4 are proved analogously to Theorems 1 and 2.

By complicating the proof, one can sharpen the estimates obtained above.

We note that a somewhat weaker assertion than Theorem 1 follows from my paper (5).

Received
15 III 1965

REFERENCES

1. I. M. Vinogradov, *Selected Works*, Publishing House of the Academy of Sciences of the USSR, 1952.

2. I. M. Vinogradov, *Proceedings of the Third All-Union Mathematical Congress*, 3, 3, 1956.

3. E. Landau, *Vorlesungen über Zahlentheorie*, 1, Leipzig, 1927.

4. Hua Loo-Keng, *The Method of Trigonometrical Sums and Its Applications in Number Theory*, Moscow, 1964.

5. A. A. Karatsuba, *Vestnik Moskov. Univ.*, Ser. Math., 1, 38 (1962).

Note: Figure translations are in progress. See original paper for figures.