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Abstract

Full Text

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ON THE APPLICATION OF THE PARAMETER-VARIATION METHOD TO THE CONSTRUCTION OF ITERATION FORMULAS OF INCREASED ACCURACY FOR DETERMINING NUMERICAL SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS*

(Presented by Academician N. N. Bogolyubov on 18 XI 1964)

The parameter-variation method proposed in works ([1,2]) is applied in the present paper to the construction of iteration formulas of increased accuracy for determining (refining) numerical solutions of nonlinear integral equations. The iteration formulas are constructed on the basis of an arbitrary quadrature formula. For estimates of the error of numerical solutions of nonlinear integral equations, see, for example, ([3,4]).

1°. Consider the nonlinear integral equation

$$\varphi(s) = \int_a^b F(s, t, \varphi(t))\, dt + f(s), \tag{1}$$

where $F(s, t, u)$ is continuous in the aggregate of variables (s, t, u), together with $F'_u(s, t, u)$, in some domain D; $f(s)$ is a continuous function on $[a, b]$.

Let an approximate numerical solution of equation (1) be given,

$$\begin{matrix} t_1 & t_2 & \cdots & t_n \\ \Phi_1^{(k)} & \Phi_2^{(k)} & \cdots & \Phi_n^{(k)}, \end{matrix} \tag{2}$$

where $a \leq t_1 < t_2 < \cdots < t_n \leq b$.

Suppose that it is required to refine the numerical solution (2) to a preassigned degree of accuracy. For this purpose, using the parameter-variation method ([1,2]), we shall construct iteration formulas of increased accuracy.

Choose some quadrature formula

$$\int_a^b \psi(t)\, dt \cong \sum_{j=1}^{n} A_j \psi(t_j) \tag{3}$$

with nodes $t_1, t_2, \ldots, t_n$—the arguments of table (2).

Denote by $x = (\Phi_1, \Phi_2, \dots, \Phi_n)$ and introduce the vector-function $G(x)$, whose components are

$$G_i(x) = G_i(\Phi_1, \Phi_2, \dots, \Phi_n) = \sum_{j=1}^{n} A_{jF}(t_i, t_j, \Phi_j), \qquad i = 1, 2, \dots, n.$$

We denote the functional matrix (Jacobian matrix) of the vector-function $G(x)$ by $J(x)$, so that

$$\{J(x)\}_{ij} = \partial G_i(\Phi_1, \Phi_2, \dots, \Phi_n)/\partial \Phi_j = A_j F'_u(t_i, t_j, \Phi_j).$$

Following the parameter-variation method ([1]), instead of equation (1) consider the vector equation

$$x - G(x) - f = \lambda, \tag{4}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a vector-parameter, $f = (f_1, f_2, \dots, f_n)$, $f_i = f(t_i)$.

* The idea of constructing iteration formulas by the parameter-variation method was reported by the author at the mathematical seminar at the I. V. Kurchatov Institute of Atomic Energy in November 1961.

We assume that equation (4) for $\lambda = 0$ has a solution $x = (\Phi_1, \Phi_2, \dots, \Phi_n)$ and that this solution is simple, i.e., the matrix $I - J(x)$ at the solution x is nonsingular.

It is obvious that the solution of equation (4) for $\lambda = 0$ is the numerical solution of equation (1) with accuracy up to the approximation (3). In other words, the solution vector $x = (\Phi_1, \Phi_2, \dots, \Phi_n)$ of equation (4) for $\lambda = 0$ will coincide, within the accuracy of approximation (3), with the vector $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ constructed from the solution $\varphi(s)$ of equation (1): $\varphi_i = \varphi(t_i)$. The approximate numerical solution vector $x^{(k)} = (\Phi_1^{(k)}, \Phi_2^{(k)}, \dots, \Phi_n^{(k)})$ of equation (1), given in the form of table (2), will be an exact solution of equation (4) for $\lambda = \lambda^{(k)}$, if $\lambda^{(k)}$ is determined by the formula

$$\lambda^{(k)} = x^{(k)} - G(x^{(k)}) - f. \tag{5}$$

We shall further assume that the solution vector x of equation (4) is a function of the components of the vector parameter $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, i.e.,

$$x = x(\lambda) = \{\Phi_1(\lambda_1, \lambda_2, \dots, \lambda_n), \dots, \Phi_n(\lambda_1, \lambda_2, \dots, \lambda_n)\}.$$

Differentiate equation (4) with respect to each λ_i $(i = 1, 2, \dots, n)$ and then solve for $\partial x/\partial \lambda_i$ $(i = 1, 2, \dots, n)$:

$$\partial x/\partial \lambda_i = D_i/D(x), \qquad i = 1, 2, \dots, n, \tag{6}$$

where $D(x)$ is the determinant of the matrix $I - J(x)$, $D_i = (D_{i1}, D_{i2}, \dots, D_{in})$, D_{ij} $(i, j = 1, 2, \dots, n)$ are the determinants obtained from the determinant $D(x)$ by replacing the elements of the j-th column by the components of the vector $\partial\lambda/\partial\lambda_i$, respectively.

The numerical values of the components of the vector $x(\lambda)$ for $\lambda = \lambda^{(k)}$ are known to us:

$$\text{for } \lambda = \lambda^{(k)} \qquad x(\lambda) = x^{(k)}. \tag{7}$$

To determine the values of the components of the vector $x(\lambda)$ for $\lambda = 0$, or, equivalently, the numerical solution of equation (1), we numerically integrate equations (6) by one of the methods of numerical integration of ordinary differential equations ([5],[6]) on the vector interval $[\lambda^{(k)}, 0]$ with initial conditions (7). In doing so, we choose the step of numerical integration $h^{(k)} = (h_1^{(k)}, h_2^{(k)}, \dots, h_n^{(k)})$ equal to

$$h^{(k)} = 0 - \lambda^{(k)} \tag{8}$$

(one may also take $h^{(k)} = -\frac{1}{m}\lambda^{(k)}$).

If the numerical values of the components of the vector $x^{(k+1)} = (\Phi_1^{(k+1)}, \Phi_2^{(k+1)}, \dots, \Phi_n^{(k+1)})$ obtained in this way still do not give, with the prescribed degree of accuracy, a solution of the homogeneous equation (4), then the process described above may be repeated, and so on $(k = 0, 1, 2, \dots)$. For each chosen method of numerical integration, one obtains its own iterative process.

Instead of equation (4), one may also consider other equations, for example the equation $x - G(x) - f = (1 - \lambda)[x^{(k)} - G(x^{(k)}) - f]$, $0 \le \lambda \le 1$, and consider differential equations of order higher than the first.

2°. **1. Euler' s method.** Suppose that Euler' s method ([6]) is chosen for the numerical integration of equations (6). According to this method, the improved numerical values of the components of the vector $x^{(k+1)} = (\Phi_1^{(k+1)}, \Phi_2^{(k+1)}, \dots, \Phi_n^{(k+1)})$ are found from the iterative formula

$$x^{(k+1)} = x^{(k)} + \Delta x^{(k)}, \qquad k = 0, 1, 2, \dots, \tag{9}$$

where the increment vector $\Delta x^{(k)}$ is determined as follows:

$$\Delta x^{(k)} = \sum_{\nu=1}^{n} \left.\frac{\partial x}{\partial \lambda_\nu}\right|_{x=x^{(k)}} h_\nu^{(k)},$$

or, equivalently, by solving the equation

$$[I - J(x^{(k)})]\Delta x^{(k)} = h^{(k)}. \tag{10}$$

leads to formula (9) and to Newton' s method.

2. The improved Euler-Cauchy method ([6]) leads to the iterative formula

$$x^{(k+1)} = x^{(k)} + \frac{1}{2}(\Delta x^{(k)} + \Delta\tilde{x}^{(k+1)}), \qquad k = 0, 1, 2, \dots, \tag{11}$$

where $\Delta x^{(k)}$ and $\Delta\tilde{x}^{(k+1)}$ are the solutions of equations (10) and (12), respectively,

$$[I - J(\tilde{x}^{(k+1)})]\Delta\tilde{x}^{(k+1)} = h^{(k)}, \qquad \tilde{x}^{(k+1)} = x^{(k)} + \Delta x^{(k)}. \tag{12}$$

3. The Runge-Kutta method ([6]). In this case the iterative formula has the form

$$x^{(k+1)} = x^{(k)} + \frac{1}{6}(l_1^{(k)} + 2l_2^{(k)} + 2l_3^{(k)} + l_4^{(k)}), \qquad k = 0, 1, 2, \dots, \tag{13}$$

where the vectors $l_i^{(k)}$ $(i = 1, 2, 3, 4)$ are the solutions of the equations

$$[I - J(x^{(k)})]l_1^{(k)} = h^{(k)}, \qquad [I - J(x^{(k)} + \tfrac{1}{2}l_1^{(k)})]l_2^{(k)} = h^{(k)},$$

$$[I - J(x^{(k)} + \tfrac{1}{2}l_2^{(k)})]l_3^{(k)} = h^{(k)}, \qquad [I - J(x^{(k)} + l_3^{(k)})]l_4^{(k)} = h^{(k)}.$$

After the numerical values Φ_i $(i = 1, 2, \dots, n)$ of the desired function $\varphi(t)$ have been determined with the prescribed degree of accuracy, by means of one or another interpolation technique one can also obtain an approximate numerical solution of equation (1) on the entire interval $[a, b]$.

Remark 1. Analogously to what has been set forth, one can construct iterative formulas of higher accuracy also for determining numerical solutions of nonlinear integral equations of the form

$$\int_a^b K(s, t, \varphi(s), \varphi(t))\, dt = f(s),$$

multidimensional nonlinear integral equations, systems of nonlinear integral equations, nonlinear differential and integro-differential equations, and others.

Remark 2. To construct iterative formulas of higher accuracy, the method of variation of a parameter can also be applied jointly with variational methods ([6,9]), the method of Yu. D. Sokolov ([7,8]), and others.

Remark 3. Iterative formulas of higher accuracy can also be constructed, using the method of variation of a parameter, for determining, generally speaking, complex solutions of systems of nonlinear equations of the form

$$f_k(x_1, x_2, \dots, x_n) = 0, \qquad k = 1, 2, \dots, n.$$

In this case the vector parameter should be taken to be complex.

3°. The method described above for constructing iterative formulas of higher accuracy can be applied directly to nonlinear integral equations, as well as to more general nonlinear functional equations.

Thus, for example, for the nonlinear integral equation

$$\varphi(s) = \int_a^b K[s, t, \varphi(t)]\, dt$$

we obtain the following iterative formulas:

a) By the Euler method for numerical integration of ordinary differential equations (cf. ([10])),

$$\varphi_{k+1}(s) = \varphi_k(s) + \Delta\varphi_k(s), \qquad k = 0, 1, 2, \dots, \tag{14}$$

where $\Delta\varphi_k(s)$ is the solution of the linear integral equation

$$\Delta\varphi_k(s) = \int_a^b K'_\varphi[s, t, \varphi_k(t)]\Delta\varphi_k(t)\, dt + \varepsilon_k(s),$$

$$\varepsilon_k(s) = \int_a^b K[s, t, \varphi_k(t)]\, dt - \varphi_k(s).$$

b) **By the improved Euler–Cauchy method**

$$\varphi_{k+1}(s) = \varphi_k(s) + \frac{1}{2}\left[\Delta\varphi_k(s) + \Delta\widetilde{\varphi}_{k+1}(s)\right], \qquad k = 0, 1, 2, \dots, \tag{15}$$

where $\Delta\varphi_k(s)$ and $\Delta\widetilde{\varphi}_{k+1}(s)$ are the solutions of the following linear integral equations, respectively,

$$\Delta\varphi_k(s) = \int_a^b K'_\varphi\left[s, t, \varphi_k(t)\right]\Delta\varphi_k(t)\, dt + \varepsilon_k(s),$$

$$\Delta\widetilde{\varphi}_{k+1}(s) = \int_a^b K'_\varphi\left[s,t,\varphi_k(t)+\Delta\varphi_k(t)\right]\Delta\widetilde{\varphi}_{k+1}(t)\,dt+\varepsilon_k(s).$$

c) **By the Runge–Kutta method**

$$\varphi_{k+1}(s) = \varphi_k(s) + \frac{1}{6}\left[\psi_k^{(1)}(s) + 2\psi_k^{(2)}(s) + 2\psi_k^{(3)}(s) + \psi_k^{(4)}(s)\right], \qquad (16)$$

$$k = 0, 1, 2, \ldots,$$

where $\psi_k^{(i)}(s)$ $(i = 1, 2, 3, 4)$ are the solutions of the following linear integral equations, respectively,

$$\psi_k^{(1)}(s) = \int_a^b K'_\varphi\left[s,t,\varphi_k(t)\right]\psi_k^{(1)}(t)\,dt+\varepsilon_k(s),$$

$$\psi_k^{(2)}(s) = \int_a^b K'_\varphi\left[s,t,\varphi_k(t)+\tfrac{1}{2}\psi_k^{(1)}(t)\right]\psi_k^{(2)}(t)\,dt+\varepsilon_k(s),$$

$$\psi_k^{(3)}(s) = \int_a^b K'_\varphi\left[s,t,\varphi_k(t)+\tfrac{1}{2}\psi_k^{(2)}(t)\right]\psi_k^{(3)}(t)\,dt+\varepsilon_k(s),$$

$$\psi_k^{(4)}(s) = \int_a^b K'_\varphi\left[s,t,\varphi_k(t)+\psi_k^{(3)}(t)\right]\psi_k^{(4)}(t)\,dt+\varepsilon_k(s).$$

Let us consider a simple example. Suppose it is required to find the positive solution of the integral equation

$$\varphi(s) = \int_0^1 4s\varphi^3(t)\,dt.$$

With the zeroth approximation $\varphi_0(s) = 0.6s$, the eighth iteration by formula (14), the fourth iteration by formula (15), and the third iteration by formula (16) give, respectively, the following results:

$$\varphi_8(s) = 1.00011819s; \qquad \varphi_4(s) = 1.00002849s; \qquad \varphi_3(s) = 0.999998054s.$$

If, however, the zeroth approximation is taken as $\varphi_0(s) = 0.55s$, then by formula (16) the fourth iteration gives

$$\varphi_4(s) = 0.999999962s.$$

The iterative processes according to formulas (14) and (15) in this case converge to another (negative) solution of the equation under consideration.

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REFERENCES

1. D. F. Davidenko, *DAN*, 88, No. 4, 601 (1953).
2. D. F. Davidenko, *Ukr. Mat. Zh.*, 5, No. 2, 196 (1953).
3. I. P. Mysovskikh, *Vestn. Leningrad Univ.*, No. 7, Ser. Math., Mech. and Astron., issue 2, 78 (1962).
4. I. P. Mysovskikh, *DAN*, 153, No. 1, 30 (1963).
5. I. S. Berezin, N. P. Zhidkov, *Methods of Computation*, 2, Moscow, 1960.
6. L. Collatz, *Numerical Methods for Solving Differential Equations*, Moscow, 1953.
7. Yu. D. Sokolov, *Ukr. Mat. Zh.*, 9, No. 4, 394 (1957).
8. Yu. D. Sokolov, *Ukr. Mat. Zh.*, 15, No. 1, 58 (1963).
9. S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Moscow, 1957.
10. L. V. Kantorovich, *Vestn. Leningrad Univ.*, No. 7, Ser. Math., Mech. and Astron., issue 2, 68 (1957).
11. D. M. Zagadskii, *DAN*, 59, No. 6, 1041 (1948).

Note: Figure translations are in progress. See original paper for figures.