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A BOUNDARY-VALUE PROBLEM OF PRESSURE FILTRATION

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Abstract

Full Text

HYDROMECHANICS

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A BOUNDARY-VALUE PROBLEM OF PRESSURE FILTRATION

(Presented by Academician P. Ya. Kochina on 12 X 1964)

This paper considers the problem of constructing the underground contour of a hydraulic-engineering structure from the epure of exit velocities.

Statement of the problem. Let

$$V = f(x) \qquad (l \leq x < \infty) \tag{1}$$

be a prescribed epure of exit velocities, where $f(x)$ is a single-valued positive monotonically decreasing function, tending to zero at infinity of order e^{-x}.

We shall assume that the boundaries of the pools AB and CD are rectilinear and located at the same level, that the water-retaining line AND is parallel to these boundaries (Fig. 1), that the sought contour is impervious, and that the foundation soil is homogeneous and isotropic; the flow obeys Darcy' s law. The filtration coefficient k and the depth of the permeable layer T are known, with $T < \infty$ (the case $T = \infty$ was considered in [1]). The head H acting on the hydraulic structure and the filtration discharge Q are prescribed in advance. It is required to construct the underground contour of the hydraulic-engineering structure.

Fig. 1
Fig. 1

Fig. 2
Fig. 2

Reduction of the solution of the problem to an integral equation. Under the assumptions made, the velocity field of the filtration flow, as is known, has a potential $\varphi = -kh$, where $h(x, y)$ is the head function. Taking on the boundary of the upstream pool the value $h = H/2$ and taking the stream function along the sought contour to be $\psi = 0$, we obtain in the plane of the complex potential $w = \varphi + i\psi$ a rectangle $ABCD$ of width kH and height Q (Fig. 2).

With the aid of the function

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$$\zeta = \text{sn}\ 2Kw/kH, \tag{2}$$

where sn is the elliptic sine, we conformally map this rectangle onto the lower half-plane $\operatorname{Im}\zeta < 0$ of the variable $\zeta = \xi + i\eta$ so that the vertices of the rectangle C, B and D, A pass respectively into the points ± 1 and $\pm 1/\lambda$ of the ξ-axis. The modulus λ is determined from the relation

$$K'/K = 2Q/kH,$$

where $K(\lambda)$ and $K' = K(\lambda')$ are complete elliptic integrals of the first kind, $\lambda' = \sqrt{1-\lambda^2}$.

From formula (2), on the segment of the ξ-axis corresponding to the boundary of the lower pool, we find

$$\psi = -\frac{kH}{2K} F\left(\arcsin\frac{\sqrt{\xi^2-1}}{\lambda'\xi}, \lambda'\right) \qquad \left(1 \le \xi \le \frac{1}{\lambda}\right), \tag{3}$$

where F is an elliptic integral of the first kind.

On the other hand, taking (1) into account, on this boundary we shall have

$$\psi = -\int_l^x f(x)\, dx \qquad (l \le x \le \infty). \tag{4}$$

Comparing (3) and (4), we obtain the dependence

$$x = \Phi(\xi) \qquad (1 \le \xi \le 1/\lambda), \tag{5}$$

where $\Phi(\xi)$ is a single-valued monotonically increasing function from l to ∞.

Suppose that on the segment $[-1, 1]$ of the ξ-axis we are given the dependence $x(\xi)$ for the unknown underground contour, with $x(1) = l$. Then, in order to determine the mapping function $z(\zeta)$, we obtain the following boundary-value problem: find in the domain $\operatorname{Im}\zeta < 0$ an analytic function, bounded at infinity and taking on the boundary of the domain the values $\operatorname{Im} z = -T$ for $1/\lambda < |\xi| < \infty$, $\operatorname{Im} z = 0$ for $1 < |\xi| < 1/\lambda$, and $\operatorname{Re} z = x(\xi)$ for $-1 \le \xi \le 1$. The solution of this problem is written in the form [2]

$$z(\zeta) = -\frac{\sqrt{\zeta^2-1}}{\pi}\int_{-1}^{1}\frac{x(\tau)\, d\tau}{(\tau-\zeta)\sqrt{1-\tau^2}} + \frac{2T}{\pi}\operatorname{ar\,th}\frac{\lambda\sqrt{\zeta^2-1}}{\lambda'}. \tag{6}$$

Extracting from this $\operatorname{Re} z$ for $\zeta = \xi$ on the segment $1 \le \xi \le 1/\lambda$ and taking formula (5) into account, we shall have

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$$-\frac{\sqrt{\xi^2-1}}{\pi}\int_{-1}^{1}\frac{x(\tau)\,d\tau}{(\tau-\xi)\sqrt{1-\tau^2}}=\frac{2T}{\pi}\operatorname{ar\,th}\frac{\lambda\sqrt{\xi^2-1}}{\lambda'}-\Phi(\xi). \tag{7}$$

This relation is a Fredholm integral equation of the first kind with a kernel specified in the rectangle $-1\le\tau\le 1,\ 1\le\xi\le 1/\lambda$. If the solution of this equation $x=x(\tau)$ $(-1\le\tau\le 1)$ is found, then from formula (6), as shown below, the equation of the sought contour is readily determined, and with relation (2) taken into account, all the necessary filtration characteristics as well.

Investigation of the integral equation obtained. Put in equation (7) $\tau=\cos\theta$ $(-1\le\tau\le 1,\ \pi\ge\theta\ge 0)$, $\xi=\operatorname{ch}\delta$ $(1\le\xi\le 1/\lambda,\ 0\le\delta\le\delta^*$, where $\delta^*=\operatorname{ar\,ch}1/\lambda)$. Then we obtain

$$\frac{\operatorname{sh}\delta}{\pi}\int_{0}^{\pi}\frac{\tilde{x}(\theta)\,d\theta}{\cos\theta-\operatorname{ch}\delta}=\frac{2T}{\pi}\operatorname{ar\,th}\frac{\lambda\operatorname{sh}\delta}{\lambda'}-\Phi(\operatorname{ch}\delta), \tag{8}$$

where $\tilde{x}(\theta)=x(\cos\theta)$.

From physical considerations it is clear that the solution of equation (8) should be sought in the class of continuous functions with bounded variation. But such a solution can always be represented by a uniformly convergent Fourier series. Thus, the required solution is naturally to be sought in the form

$$\tilde{x}(\theta)=\sum_{n=1}^{\infty}a_n\cos n\theta. \tag{9}$$

Then the solution of equation (8) in this class will be equivalent to the solution of the equation

$$\Phi(\operatorname{ch}\delta)=\frac{2T}{\pi}\operatorname{ar\,th}\frac{\lambda\operatorname{sh}\delta}{\lambda'}+\sum_{n=1}^{\infty}a_n e^{-n\delta}\qquad(0\le\delta\le\delta^*). \tag{10}$$

Thus, if the function $\Phi(\operatorname{ch}\delta)$ is representable in the form (10) and, moreover, the series (9) converges uniformly to a function with bounded variation,

then the solution of equation (8) exists, is unique, and is expressed by the series (9). In particular, the solution exists for all functions of the form (10) with a finite series.

In numerical calculations, when the function $\Phi(\operatorname{ch}\delta)$ is given in tabular form, the coefficients a_n $(n=1,2,\ldots)$ can be approximately determined from the system of linear algebraic equations

$$\sum_{n=1}^{N}a_n e^{-n\delta_j}=\Phi(\operatorname{ch}\delta_j)-B(\delta_j)\quad(j=1,2,\ldots,N), \tag{11}$$

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where

$$B(\delta_j) = \frac{2T}{\pi}\operatorname{ar\,th}\frac{\lambda\operatorname{sh}\delta_j}{\lambda'}, \qquad \delta_j \in [0, \delta^*].$$

It is not difficult to see that this system is uniquely solvable, since its determinant (a Vandermonde determinant) is different from zero. Having computed the coefficients a_n and substituted them into formula (9), we find the dependence $x = x(\cos\theta)$ needed for the subsequent solution of the problem.

Formulas for constructing the underground contour and determining the filtration characteristics. Let $x = x(\xi)$ be the solution of the integral equation (7). Carrying out in formula (6) the limiting transition as $\zeta \to \xi$ $(-1 \le \xi \le 1)$, we obtain the parametric equations of the required underground contour

$$x = x(\xi),$$

$$y = \frac{\sqrt{1-\xi^2}}{\pi}\int_{-1}^{1}\frac{x(\tau)\,d\tau}{(\tau-\xi)\sqrt{1-\tau^2}} - \frac{2T}{\pi}\operatorname{arc\,tg}\frac{\lambda\sqrt{1-\xi^2}}{\lambda'} \quad (-1 \le \xi \le 1), \quad (12)$$

where the integral is to be understood in the sense of the Cauchy principal value.

If we set $\xi = \cos\gamma$, $\tau = \cos\theta$ $(-1 \le \xi, \tau \le 1,\ \pi \ge \gamma, \theta \ge 0)$ and use the representation (9), then equations (12) may be written in the form

$$\left.\begin{aligned} \tilde{x}(\gamma) &= \sum_{n=1}^{\infty} a_n \cos n\gamma, \\ \tilde{y}(\gamma) &= \sum_{n=1}^{\infty} a_n \sin n\gamma - \frac{2T}{\pi}\operatorname{arc\,tg}\frac{\lambda\sin\gamma}{\lambda'} \end{aligned}\right\} \quad (0 \le \gamma \le \pi). \qquad (13)$$

Having constructed the contour, it is not difficult to find the required filtration characteristics of the groundwater flow. Thus, for example, determining from formula (2) the head function

$$h = -\frac{H}{2K}F(\arcsin\xi, \lambda) \quad (-1 \le \xi \le 1)$$

and using the first of relations (12), we construct the epure of filtration pressure $h = h(x)$ $(x_0 \le x \le l)$, where $x_0 = x(-1)$ is the abscissa of the initial point of the flow outlet B (Fig. 1).

Taking (2) and (6) into account, we find the complex gradient of filtration in the form

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$$I_x - iI_y = \frac{H}{2iK}\left[\sqrt{1-\lambda^2\xi^2}\,p(\xi) + \frac{2T\lambda\lambda'\xi}{\pi\sqrt{1-\lambda^2\xi^2}}\right]^{-1},$$

where

$$p(\xi) = \frac{1}{\pi}\int_{-1}^{1}\frac{dx}{d\tau}\frac{\sqrt{1-\tau^2}}{\tau-\xi}\,d\tau.$$

Separating in formula (6) $\operatorname{Re} z(\zeta)\big|_{\zeta=\xi} = \Phi^*(\xi)$ on the interval $-1/\lambda \le \xi \le -1$ and requiring that the function $\Phi^*(\xi)$ be monotonically increasing (the necessity of this follows from physical considerations), we obtain the solvability condition for our problem

$$2T\lambda\lambda'/\pi > M, \tag{14}$$

where M is the maximum of the function $(1/\xi)(\lambda^2\xi_0^2 - 1)p(\xi)$ in the interval $(-1/\lambda, -1)$. It can be shown [2] that if the found dependence $x(\xi)$ increases monotonically for $-1 \leqslant \xi \leqslant 1$, then condition (14) takes the form

$$\frac{\lambda}{\lambda'} \geqslant \frac{\pi}{2T}p(-1).$$

Solving it with respect to λ, we obtain

$$\lambda \geqslant \pi p(-1)\,[\pi^2 p^2(-1) + 4T^2]^{-1/2}.$$

Consequently, the solution will have physical meaning only when the discharge Q does not exceed a certain admissible value.

The case of an unbounded depth of the water-permeable layer. Since the length of the underground contour L is assumed to be different from zero, we shall have $Q = \infty$, $\lambda = 0$. Formulas (2) and (3) accordingly take the form

$$\xi = \sin\frac{\pi w}{kH}, \qquad \psi = -\frac{kH}{\pi}\operatorname{ar\,ch}\xi \qquad (1 \leqslant \xi < \infty).$$

Taking into account that, for a sufficiently arbitrary outline of the underground contour,

$$\lim_{T\to\infty}\lambda T = \frac{1}{2}\pi A,$$

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where A is a material constant different from zero [2], in formula (6), instead of the last term, we obtain $A\sqrt{\xi^2-1}$. Equation (10) is written in the form

$$\Phi(\operatorname{ch}\delta) = \frac{A}{2}e^{\delta} + \left(a_1 - \frac{A}{2}\right)e^{-\delta} + \sum_{n=2}^{\infty} a_n e^{-n\delta} \qquad (0 \leqslant \delta < \infty),$$

whence, as is easy to see, the necessary condition for the existence of a solution will be

$$0 < A = \lim_{\delta\to\infty} \frac{2\Phi(\operatorname{ch}\delta)}{e^{\delta}} < \infty.$$

At the same time this condition serves to determine the constant A.

In system (11), instead of $B(\delta_j)$, one should write $A\operatorname{sh}\delta_j$. The second of relations (13) takes the form

$$\tilde{y}(\gamma) = \sum_{n=1}^{\infty} a_n \sin n\gamma - A\sin\gamma.$$

The remaining formulas are simplified accordingly.

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References

1. N. B. Il' inskii, *Proceedings of the Seminar on Inverse Boundary-Value Problems*, issue 2, Kazan, 1964, p. 19.

2. M. T. Nuzhin, N. B. Il' inskii, *Methods for Constructing an Underground Contour of Hydraulic Structures; Inverse Boundary-Value Problems in Filtration Theory*, Kazan, 1963.

Note: Figure translations are in progress. See original paper for figures.

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