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1.** In the present paper the following problem is considered:

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Abstract

Full Text

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SOME BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF ELLIPTIC EQUATIONS OF SECOND ORDER THAT DO NOT SATISFY THE CONDITION OF YA. B. LOPATINSKII

(Presented by Academician M. A. Lavrent'ev on 10 VIII 1964)

1. In the present paper the following problem is considered:
In a bounded plane simply connected domain D with boundary Γ , it is required to find a regular solution of the elliptic system

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0, \quad (1)$$

belonging to the class $C_\alpha^2(\bar{D})$ and satisfying the boundary condition

$$au_x + bu_y + cu = f \quad \text{on } \Gamma, \quad (2)$$

where $u = (u_1, u_2)$ is the unknown vector, $f = (f_1, f_2)$ is a given real vector on Γ ; A, B, C, a, b, c are real constant square matrices of second order.

Let $z = t(s)$ ($z = x + iy$) be a parametric equation of the boundary Γ , where s is arc length. It is assumed that $f \in C_\alpha^2(\Gamma)$, $t(s) \in C_\alpha^3(\Gamma)$.

In paper ⁽¹⁾ it was proved that the Ya. B. Lopatinskii condition ensures the Noether property of problem (1)–(2). In the present paper a more general condition is indicated under which problem (1)–(2) in any simply connected domain is Noetherian. If this condition is not fulfilled, then one can always indicate a domain with an arbitrarily smooth boundary where problem (1)–(2) is not Noetherian even in the case when f is an infinitely differentiable function.

We shall call problem (1)–(2) Noetherian if the homogeneous problem (1)–(2) has a finite number of linearly independent solutions, and the inhomogeneous problem is solvable when the function f is subject to a finite number of orthogonality conditions.

2. We first consider the case where the characteristic equation

$$|A + 2B\lambda + C\lambda^2| = 0 \quad (3)$$

has multiple roots. Let λ_0 be a multiple root of equation (3) with positive imaginary part. Without loss of generality we shall assume that $\lambda_0 = i$. This can always be achieved by means of a transformation of the independent variables of the form $X = x + \operatorname{Re} \lambda_0 y$, $Y = \operatorname{Im} \lambda_0 y$.

In this case the general solution of system (1) in the domain D is given by the formula (see ⁽²⁾, p. 64)

$$u = \operatorname{Re} \left[\alpha_1 \varphi_1(z) + \alpha_2 \left(\varphi_2(z) + kz \overline{\varphi_1'(z)} \right) \right], \quad (4)$$

where $\varphi_1(z)$ and $\varphi_2(z)$ are arbitrary analytic functions in the domain D , with $k = 1$, while α_1 and α_2 ($\alpha_2 \neq 0$) are two-dimensional constant vectors that are particular solutions of the system of equations

$$(A + 2Bi - C)\alpha_2 = 0, \quad (A + 2Bi - C)\alpha_1 + 2(A + C)\alpha_2 = 0. \quad (5)$$

if the rank of the matrix $A + 2Bi - C$ is equal to 1, and $k = 0$, $\alpha_1 = (1, 0)$, $\alpha_2 = (0, 1)$, if all elements of the matrix $A + 2Bi - C$ are zero.

We note that the system of equations (5) always has solutions α_1 and α_2 , where $\alpha_2 \neq 0$. Suppose that the origin of coordinates lies inside the domain D . On the functions $\varphi_1(z)$ and $\varphi_2(z)$ one may impose the conditions

$$\operatorname{Im}[(\alpha_1 - 2k\alpha_2)\varphi_1(0) + \alpha_2\varphi_2(0)] = 0.$$

Under these conditions the analytic functions φ_1 and φ_2 are determined by $u(x, y)$ uniquely.

Substituting the general solution (4) into the boundary condition (2), we obtain

$$\begin{aligned} \operatorname{Re}[\beta_1 \varphi_1(z) + \beta_2(\varphi_2(z) + k\bar{z}\varphi_1'(z)) + \beta_3 \varphi_1'(z) + \\ + \beta_4(\varphi_2'(z) + k\bar{z}\varphi_1''(z))] = f \quad \text{on } \Gamma, \end{aligned} \quad (6)$$

where

$$\beta_1 = c\alpha_1, \quad \beta_2 = c\alpha_2, \quad \beta_3 = (a + bi)\alpha_1 + k(a - bi)\alpha_2, \quad \beta_4 = (a + bi)\alpha_2.$$

Let one of the following conditions be satisfied:

- a) $\beta_3 = 0$, $\beta_4 = 0$; the vectors β_1 and β_2 are linearly independent.
- b) $\beta_4 = 0$, $\beta_3 \neq 0$; the vectors β_2 and β_3 are linearly independent.
- c) $\beta_4 = 0$, $\beta_3 \neq 0$; the vectors β_2 and β_3 are linearly dependent, and the vectors β_1 and β_2 are linearly independent.
- d) $\beta_4 \neq 0$, $k = 0$; the vectors β_3 and β_4 are linearly dependent, β_2 and β_4 are linearly independent; $|k_2 - l_1| + |l_2| \neq 0$, where k_2, l_1 , and l_2 are numbers determined from the equations

$$\beta_3 = k_2\beta_4, \quad \beta_1 = l_1\beta_2 + l_2\beta_4. \quad (7)$$

- e) $\beta_4 \neq 0$, $k = 1$; the vectors β_3 and β_4 are linearly dependent, β_2 and β_4 are linearly independent; $|k_2 - l_1| \neq 0$, where k_2 and l_1 are determined from (7).
- f) $\beta_4 \neq 0$, β_3 and β_4 are linearly dependent, β_1 and β_4 are linearly independent, β_2 and β_4 are linearly dependent.
- g) β_3 and β_4 are linearly independent.

When any one of these conditions is fulfilled, problem (1)–(2) is reduced to an equivalent singular integral equation of normal type, if one uses, respectively for cases a)–g), the integral representations of the analytic functions $\varphi_1(z)$ and $\varphi_2(z)$ in the form

$$1) \quad \varphi_1(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\mu_1(t) dt}{t - z} + ic_1,$$

$$\varphi_2(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\mu_2(t) - k\bar{t}\mu_1'(t)}{t - z} dt + ic_2.$$

Here and in what follows $\mu_1(t)$ and $\mu_2(t)$ are real functions; c_1 and c_2 are real constants, i is the imaginary unit;

$$\mu'(t) = \frac{d\mu(t)}{dt}, \quad t = \xi + i\eta \in \Gamma, \quad \bar{t} = \xi - i\eta, \quad \bar{t}' = \frac{d\bar{t}}{dt}, \quad d_j = \int_{\Gamma} \mu_j(t) ds.$$

$$2) \quad \varphi_1(z) = \int_{\Gamma} \mu_1(t) \ln \left(1 - \frac{z}{t} \right) ds + d_1 + ic_1,$$

$$\varphi_2(z) = \int_{\Gamma} \frac{\mu_2(t) + k\bar{t}\mu_1'(t)\bar{t}'_s}{t - z} dt + c_2.$$

$$3) \quad \varphi_1(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\mu_1(t) dt}{t - z} + ic_1,$$

$$\varphi_2(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\mu_2(t) - \mu_1'(t)(k_1 + k\bar{t})}{t - z} dt + ic_2,$$

where k_1 is determined from the equality $\alpha_3 = k_1\alpha_2$, $\bar{t}'_s = d\bar{t}/ds$.

4)

$$l_1\varphi_1(z) + \varphi_2(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\mu_2(t) dt}{t - z} + ic_1,$$

$$\varphi_1(z) = \int_{\Gamma} \left[\mu_1(t) + \frac{1}{\pi i(k_2 - l_1)} \mu_2'(t)t'_s \right] \ln \left(1 - \frac{z}{t} \right) ds + d_1 + ic_2,$$

if $k_2 - l_1 \neq 0$, and

$$l_1 \varphi_1(z) + \varphi_2(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\mu_2(t) dt}{t - z} + ic_1,$$

$$l_2 \varphi_1(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{(\mu_1(t) - \mu_2'(t)) dt}{t - z} + ic_2,$$

if $k_2 - l_1 = 0$.

5)

$$\varphi_1(z) = \int_{\Gamma} \frac{\mu_1(t) dt}{t - z} + c_1,$$

$$\begin{aligned} \varphi_2(z) = & \int_{\Gamma} (\mu_2(t) \bar{t}'_s + k \bar{t} \mu_1''(t) + k_2 \mu_1'(t) + l_2 \mu_1(t)) \times \\ & \times \ln \left(1 - \frac{z}{t} \right) dt + d_2 + ic_2, \end{aligned}$$

where k_2 and l_2 are determined from (7).

6)

$$\varphi_1(z) = \int_{\Gamma} \frac{\mu_1(t) dt}{t - z} + ic_1,$$

$$\varphi_2(z) = \int_{\Gamma} (\mu_2(t) \bar{t}'_s + k \bar{t} \mu_1''(t) + k_2 \mu_1'(t)) \ln \left(1 - \frac{z}{t} \right) dt + d_2 + ic_2.$$

7)

$$\varphi_1(z) = \int_{\Gamma} \mu_1(t) \ln \left(1 - \frac{z}{t} \right) ds + d_1 + ic_1,$$

$$\varphi_2(z) = \int_{\Gamma} (\mu_2(t) + k \bar{t} \mu_1'(t)) \ln \left(1 - \frac{z}{t} \right) ds + d_2 + ic_2.$$

In these integral representations the quantities μ_1, μ_2, c_1, c_2 are determined uniquely by $\varphi_1(z)$ and $\varphi_2(z)$. The integral representations of analytic functions $\varphi_1(z)$ and $\varphi_2(z)$ used here are simple consequences of the integral representations given in (3). Analytic integral representations of analytic functions were used in the paper (4).

After such an equivalent reduction, by virtue of the known theorems of the theory of singular integral equations (see (3)), we obtain:

Theorem 1. *If the characteristic equation (3) has a multiple root i and one of the conditions a)–) is satisfied, then problem (1)–(2) is Noetherian. The index of problem (1)–(2) is equal to zero if one of the conditions a),) is satisfied; is equal to 2 if one of the conditions),) is satisfied; is equal to 4 if condition) is satisfied. If condition) is satisfied, the index of problem (1)–(2) is equal to zero if $k_2 - l_1 = 0$, and is equal to 2 if $k_2 \neq l_1$. If condition) is satisfied, the index of problem (1)–(2) is equal to 2 if $|k_2 - l_1| > 1$, and is equal to 6 if $|k_2 - l_1| < 1$.*

Let us note that the Ya. B. Lopatinskii condition is equivalent to condition).

3. Let the characteristic equation (3) have simple roots. Denote by λ_1 and λ_2 the roots of this equation with positive imaginary parts. The general solution of system (1) in this case can be written in the form (see (2))

$$u = \operatorname{Re}(\gamma_1 \varphi_1(z_1) + \gamma_2 \varphi_2(z_2)), \quad (8)$$

where γ_1 and γ_2 are two-dimensional vectors which are nontrivial particular solutions of the equations

$$A + 2B\lambda_k + C\lambda_k^2 \quad (k = 1, 2);$$

$z_1 = x + \lambda_1 y$, $z_2 = x + \lambda_2 y$; $\varphi_1(z)$ and $\varphi_2(z)$ are arbitrary analytic functions, respectively, in the domains D_1 and D_2 . The domains D_1 and D_2 are obtained from the domain D by means of the mappings $\zeta = x + \lambda_1 y$ and $\zeta = x + \lambda_2 y$.

The functions $\varphi_1(z)$ and $\varphi_2(z)$ at the zero point can be made subject to the conditions $\operatorname{Re}(\lambda_1 \varphi_1(0) + \lambda_2 \varphi_2(0))$. Under these conditions $\varphi_1(z)$ and $\varphi_2(z)$ will be determined uniquely with the aid of $u(x, y)$.

Substituting the general solution (8) into the boundary condition (2), we obtain

$$\operatorname{Re}(\delta_1 \varphi_1(z_1) + \delta_2 \varphi_2(z_2) + \delta_3 \varphi_1'(z) + \delta_4 \varphi_2'(z)) = f \text{ on } \Gamma, \quad (9)$$

where $\varphi_j'(z) = d\varphi_j(z)/dz$, $\delta_1 = c\gamma_1$, $\delta_2 = c\gamma_2$, $\delta_3 = (a + \lambda_1 b)\gamma_1$, $\delta_4 = (a + \lambda_2 b)\gamma_2$. If one of the vectors δ_3 and δ_4 is different from zero, we shall assume that $\delta_3 \neq 0$ (this does not restrict generality).

Let one of the following conditions be fulfilled:

- a*) $\delta_3 = 0$, $\delta_4 = 0$; the vectors δ_1 and δ_2 are linearly independent.
- b*) $\delta_3 \neq 0$, δ_3 and δ_4 are linearly dependent, while the vectors δ_2 and δ_3 are linearly independent,

$$|(1 - \lambda_1 i)l_3 k_3 - (1 - \lambda_2 i)| \neq |(1 + \lambda_1 i)l_3 k_3 - (1 + \lambda_2 i)|,$$

where k_3 and l_3 are numbers determined from the equalities $\delta_4 = k_3 \delta_3$, $\delta_1 = l_3 \delta_2 + l_4 \delta_3$.

c*) $\delta_3 \neq 0$, δ_3 and δ_4 are linearly dependent, δ_2 and δ_3 are linearly dependent, δ_1 and δ_3 are linearly independent, $|k_3| + |k_4| \neq 0$, where k_4 and k_3 are determined from the equalities $\delta_4 = k_3 \delta_3$, $\delta_2 = k_4 \delta_3$.

d*) δ_3 and δ_4 are linearly independent.

Analogously to the case of multiple roots, in this case problem (1)–(2) also reduces to an equivalent one-dimensional singular integral equation of normal type, and the following theorem is proved.

Theorem 2. *If the characteristic equation (3) has simple roots and one of the conditions a*)–d*) is satisfied, then the homogeneous problem (1)–(2) in the class $C'_\alpha(\bar{D})$ has a finite number of linearly independent solutions, while the nonhomogeneous problem in the same class has a solution for $f \in C'_\alpha(\Gamma)$, provided the function f is subject to a finite number of orthogonality conditions. The index of this problem is equal to zero if condition a*) is satisfied and is equal to 4 if condition d*) is satisfied. When condition b*) is satisfied, the index of problem (1)–(2) is equal to 2 if the inequality*

$$|(1 - \lambda_1 i)l_3 k_3 - (1 - \lambda_2 i)| > |(1 + \lambda_2 i)l_3 k_3 - (1 + \lambda_2 i)|$$

holds, and is equal to 6 if the reverse inequality holds. When condition c) is satisfied, the index of problem (1)–(2) is equal to zero if $k_3 = 0$, and is equal to 2 if $k_3 \neq 0$. In this case the Ya. B. Lopatinskii condition is equivalent to condition d*).*

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CITED LITERATURE

1. A. I. Volpert, *Trudy Moskovsk. Matemat. Obshch.*, **10**, 41 (1961).
2. A. V. Bitsadze, *Equations of Mixed Type*. Moscow, 1959.
3. N. I. Muskhelishvili, *Singular Integral Equations*. Moscow, 1946.
4. E. V. Zolotareva, *DAN*, **132**, No. 4 (1960).

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