Reports of the Academy of Sciences of the USSR

1965

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Volume 163, No. 3

MATHEMATICS

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ON REFLEXIVE SPACES GENERALIZING REFLEXIVE ORLICH SPACES

(Presented by Academician L. V. Kantorovich, January 14, 1965)

The present work is a continuation of the author' s paper (^{1}). First we shall give some simple and, nevertheless, apparently previously unknown results on N-functions. Then, relying on these results, we shall prove several theorems on reflexive* Banach spaces that are a generalization of the known Orlicz spaces.

We shall use the terminology and notation of the monographs (^{2},^{3}). Let us recall some definitions.

1. A continuous convex function $M(u)$ is called an N-function (^{2}) if it is even and satisfies the conditions

$$M(u) > 0 \text{ for } u > 0, \qquad \lim_{u\to 0} \frac{M(u)}{u} = 0, \qquad \lim_{u\to\infty} \frac{M(u)}{u} = \infty.$$

Two N-functions M and N are called complementary to each other if their derivatives $p(u) = M'(u)$ and $q(v) = N'(v)$ are related by

$$q(v) = \sup_{p(u)\le v} u, \qquad p(u) = \sup_{q(v)\le u} v.$$

2. Two N-functions M and N are called equivalent for all u if there exist constants $a, b > 0$ such that, for all $u \ge 0$, the inequality

$$M(au) \le N(u) \le M(bu). \tag{1}$$

holds.

If inequality (1) is satisfied only for $u \ge u_0 > 0$, then $M(u)$ and $N(u)$ are said to be equivalent for large u.

3. An important role in many questions is played by the Δ_2-condition. An N-function M satisfies the Δ_2-condition for all u if there exists a constant k such that, for all $u > 0$, the inequality

$$M(2u) \leq kM(u). \tag{2}$$

holds.

If (2) is satisfied only for $u \geq u_0 > 0$, then M is said to satisfy the Δ_2-condition for large u.

Let us recall that, in order that the N-function complementary to the N-function M satisfy the Δ_2-condition for all (for large) values of the argument, it is necessary and sufficient that

$$M(u) \leq \frac{1}{2l} M(lu) \tag{3}$$

for all (for large) values of u, where $l > 1$ (²).

Introduce the following notation. $\mathfrak{M}_2$ is the class of all N-functions satisfying the Δ_2-condition for all u; $\mathfrak{M}_2^2$ is the class of all N-functions satisfying the Δ_2-condition together with their complementary functions for all values of the argument; $\mathfrak{N}_2$ is the class of all N-functions satisfying the Δ_2-condition for large u; $\mathfrak{N}_2^2$ is the class of all N-functions satisfying the Δ_2-condition together with their complementary functions, for large values of the argument.

Theorem 1. Let $M \in \mathfrak{N}_2$. In order that $M \in \mathfrak{N}_2^2$, it is necessary and sufficient that either of the following two conditions hold:

1) There exist $P \in \mathfrak{N}_2$ and a number $p > 1$ such that $M(u)$ is equivalent to $P(|u|^p)$ for large u.
2) There exist $Q \in \mathfrak{N}_2$ and a number $q > 1$ such that $M(u)$ is equivalent to $[Q(u)]^q$ for large u.

Theorem 1′**.** Let $M \in \mathfrak{N}_2$. In order that $M \in \mathfrak{N}_2^2$, it is necessary and sufficient that either of the following two conditions hold:

1) There exist $P \in \mathfrak{N}_2$ and a number $p > 1$ such that $M(u)$ is equivalent to $P(|u|^p)$ for all u.
2) There exist $Q \in \mathfrak{N}_2$ and a number $q > 1$ such that $M(u)$ is equivalent to $[Q(u)]^q$ for all u.

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* Throughout this note the term reflexivity is understood in the sense of the theory of normed spaces.

Remark. In the necessity part these theorems can be somewhat strengthened, namely, one may require that $P, Q \in \mathfrak{N}_2^2$ in Theorem 1′ and $P, Q \in \mathfrak{N}_2^2$ in Theorem 1.

Let us give a brief outline of the proof, for example, of Theorem 1′. In the sufficiency part the theorem is easily verified. The necessity of condition 1) can be shown, for example, as follows. Put

$$P(u) = \inf \left\{ \sum_{i=1}^{3} \alpha_i M\left(u_i^{1/p}\right) \right\},$$

where the infimum on the right-hand side is taken over all possible sets of numbers $\alpha_1, \alpha_2, \alpha_3, u_1, u_2, u_3 \geq 0$ satisfying the conditions

$$\sum_{i=1}^{3} \alpha_i = 1, \qquad \sum_{i=1}^{3} \alpha_i u_i = |u|.$$

For p one may take any number satisfying the inequality

$$1 < p < 1 + \ln 2 / \ln l,$$

where l is from inequality (3). With the aid of Carathéodory' s theorem stating that the convex hull of a set situated in an n-dimensional linear space coincides with all possible convex combinations of all possible $(n+1)$-point sets from this set (^{4}), we are convinced that the function $P(u)$ is convex. Then, by means of simple estimates, we verify that $P(u) \in \mathfrak{N}_2$ (even $P(u) \in \mathfrak{N}_2^2$) and that $M(u)$ is equivalent to $P(|u|^p)$ for all u.

Next we shall need some facts from the theory of linearly partially ordered spaces, in particular the notion of a KB-space introduced by L. V. Kantorovich. A KB-space is a K-space in which a norm is defined that turns it into a Banach space, and moreover the following conditions are satisfied: 1) from $|x| \leq |y|$ it follows that $\|x\| \leq \|y\|$ (monotonicity of the norm); 2) if $x_n \downarrow 0$, then $\|x_n\| \to 0$; 3) if $x_n \uparrow +\infty$, then $\|x_n\| \to +\infty$ (see also (^{3})).

Let now X be an arbitrary KB-space, $\hat{X}$ the maximal extension of X (^{3}), with the unit 1 fixed in X. It makes sense (^{5}) to speak of continuous functions defined on $\hat{X}$. Put, for an arbitrary number $p > 1$,

$$X_p = \{x : x \in \hat{X},\ |x|^p \in X\},$$

i.e., X_p consists of all elements of $\hat{X}$ whose p-th power of the modulus is contained in X. Introduce on X_p a norm by setting

$$\|x\|_p = \||x|^p\|^{1/p},$$

where $\| \ \|$ is the norm in the original KB-space X. Thus the space $(X_p, \| \ \|_p)$ so obtained, or any space algebraically and structurally isomorphic to it, will be called a space of type X_p.

In (^{1}) it was noted that for an arbitrary KB-space X and number $p > 1$, $(X_p, \| \ \|_p)$ is a reflexive KB-space. If X is taken to be the usual space L_T of all summable functions on a certain measure space T, then X_p coincides with the usual space L_T^p.

Theorem 2*. Let X be a KB-space with a unit, $M(u) \in \mathfrak{N}_2^2$. Put

$$X_M = \{x : \ x \in X, \ M(x) \in X\}$$

and, for $x \in X_M$,

$$\|x\|_M = \inf\{K : \ K > 0, \ \|M(x/k)\| \leq 1\},$$

where $\| \ \|$ is the norm in X.

Then $(X_M, \| \ \|_M)$ is a reflexive KB-space.

Theorem 2′. Let $\widehat{X}$ be a KB-space, X its maximal extension, in which the unit 1 is fixed, $M(u) \in \mathfrak{N}_2^2$. Put:

$$\widehat{X}_M = \{x : \ x \in \widehat{X}, \ M(x) \in X\}$$

and, for $x \in X_M$,

$$\|x\|_M = \inf\{k : \ k > 0, \ \|M(x/k)\| \leq 1\},$$

where $\| \ \|$ is the norm in X.

Then $(\widehat{X}_M, \| \ \|_M)$ is a reflexive KB-space.

Let us prove, for example, Theorem 2′. According to Theorem 1′, there exist $P(u) \in \mathfrak{M}_2$ and a number $p > 1$ such that $M(u)$ is equivalent to $P(|u|^p)$ for all u. Then it is not difficult to show that the spaces X_M and $(X_P)_p$ coincide in their stock of elements, and since both of them are KB-lineals (even KB-spaces), their norms are equivalent. It remains to observe that $(X_P)_p$ is a space of type X_p, if X_P is taken for X, and therefore, by Theorem 1 of (^{1}), is reflexive.

Remark 1. It is known that the Orlicz space constructed from an N-function $M(u)$ on a certain measure space T is reflexive if and only if $M(u) \in \mathfrak{N}_2^2$, if the measure T is finite, and when $M(u) \in \mathfrak{M}_2^2$, if the measure T is infinite (^{2,6}). Thus our Theorems 2 and 2′ are generalizations of the indicated results in the sufficiency part, which is obtained if in Theorems 2 and 2′ one puts $X = L_T$, with in the first case the measure T finite, and in the second infinite.

Remark 2. From all that has been said it follows that a reflexive Orlicz space is always equivalent to a space of type X_p with some $p > 1$. We note that in (^{1}) an example is given of a reflexive KB-space not algebraically and structurally isomorphic to a space X_p for any KB-space X and number $p > 1$. Thus, in this sense, an arbitrary reflexive KB-space is more complicated in structure than a reflexive Orlicz space.

Remark 3. In contrast to Orlicz spaces, the case is possible when the space X_M, constructed from a KB-space X and an N-function $M(u)$, is reflexive, but $M \notin \mathfrak{N}_2^2$. For example, if one puts $X = L^p[0;1]$ for any $p > 1$, then X_M is reflexive if and only if $M(u) \in \mathfrak{N}_2$.

We also note the following theorem, which we state for the simplest case $T = [0;1]$.

Theorem 3. The Orlicz space $L^*_M[0;1]$ is reflexive if and only if there exist $Q(u) \in \mathfrak{N}_2$ and a number $q > 1$ such that the spaces L^*_M and $(L^1)_Q$ are isomorphic algebraically, topologically, and structurally.

Proof of this theorem follows easily from Theorem 1. From Theorem 3 it follows that, in a certain sense, reflexive Orlicz spaces are related to the spaces L^p for $p > 1$ in the same way as arbitrary Orlicz spaces are related to the space L.

Let now X be a KB-space which is a foundation in $L[0;1]$ and contains all bounded functions from $L[0;1]$, $p > 1$ arbitrary—

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* In (^{1}) this theorem is given only for the case when X is continuous and separable.

number, $p_1 = p/(p-1)$. Put

$$X' = \{y:\ y \in L,\ xy \in L \text{ for every } x \in X\},$$

$$(X_p)' = \{y:\ y \in L,\ xy \in L \text{ for every } x \in X_p\},$$

$$(X')_p = \{x:\ x \in L,\ |x|^p \in X'\}.$$

Recall that X' and $(X_p)'$ are naturally identified with the spaces conjugate to X and X_p, respectively $^{(1)}$. Put also

$$(X')_p \times L^{p_1} = \{xy:\ x \in (X')_p,\ y \in L^{p_1}\}.$$

Theorem 4. The equality

$$(X_p)' = (X')^p \times L^{p_1}.$$

holds.

Remark. The indicated formula is a generalization of the well-known fact that, in a certain sense, the space L^{p_1} is conjugate to the space L^p.

The author expresses gratitude to his scientific adviser Prof. B. Z. Vulikh for his attention to the present work.

Note added in proof. After the paper had been submitted for publication, the author learned that results close in content to Theorems 1 and 1′ are contained in (7).

Received
6 I 1965

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Note: Figure translations are in progress. See original paper for figures.

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