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**Abstract**

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# RECURRENT AND ALMOST PERIODIC LIMIT TRAJECTORIES OF NONAUTONOMOUS SYSTEMS OF DIFFERENTIAL EQUATIONS

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One of the basic facts of the topological theory of dynamical systems is G. D. Birkhoff' s theorem that in every compact dynamical system there exists a recurrent motion (see (^1)). If, moreover, the system is stable in the sense of Lyapunov, then this motion is almost periodic. For nonautonomous systems this is no longer true. However, V. V. Nemytskii stated the following hypothesis. If the set of solutions of the system is supplemented in a reasonable way (it is assumed that it has a compact invariant set) by certain "limit" trajectories, then in this enlargement there will already be a recurrent trajectory $x(t)$, and if an additional condition of Lyapunov-stability type is imposed, then $x(t)$ is almost periodic.

In the present paper the corresponding definitions are introduced and theorems confirming V. V. Nemytskii' s hypothesis are proved. It is then shown that, instead of the existence of an invariant compact set $R$, it is sufficient that there exist a solution $x(t)$ such that $x(t) \in R$ for $t \geq t_0$.

Consider the equation

$$\frac{dx}{dt} = f(x,t) \tag{1}$$

in a separable locally convex space $L$. (In particular, in the $n$-dimensional space $E^n$.)

We first assume that $f(x,t)$ is bounded (i.e., the set of its values is bounded). At the end of the note we shall clarify the meaning of this boundedness and eliminate it. We assume $f(x,t)$ to be continuous on $R \times E^1$, where the compact set $R \subset L$ is such that for every $x_0 \in R$ and every $t_0$ there exists a solution $x(t)$ of (1) such that $x(t_0) = x_0$ and $x(t) \in R$ for all $t$. (No other assumptions, for example uniqueness of $x(t)$, are required.)

Denote by $E_0$ the set of functions $x(t)$—solutions of (1) with values in $R$. Let

$$E_1 = \bigcup_{-\infty<h<+\infty} T_h E_0,$$

where $T_h$ is the operator of shifting functions to the left by $h$. Let $\mathcal{L}$ be the space of uniform convergence on compact sets of continuous functions $x(t)$ with values in $L$, and let $E$ be the closure of $E_1$ in $\mathcal{L}$. It is obvious that if $x(t) \in E$, then for every $t$, $x(t) \in R$.

**Definition 1.** We shall call $E$ an **enlargement of the set of solutions of equation** (1) **in the compact set** $R$. Functions $x(t) \in E \setminus E_1$ will be called **limit solutions of equation** (1).

**Definition 2.** We shall call $R_1 \subseteq R$ **invariant** $(E)$ if for every $x_0 \in R_1$ and every $t_0$ there is a function $x(t) \in E$ such that $x(t_0) = x_0$, $x(t) \in R_1$ for all $t$. An invariant $(E)$ compact set $R_1 \neq \emptyset$ is called a **minimal** $(E)$ **set** if no nonempty compact set $R_2 \subset R_1$ (proper inclusion) is invariant $(E)$.

**Lemma 1.** *There exists a minimal* $(E)$ *set* $\Sigma \subseteq R$.

**Proof.** See (^1), p. 401, proof of Theorem 26, with the difference that since the weight $\tau$ of the space $R \subset L$ is no longer necessarily countable, the break of the transfinite sequence constructed there is also ···

takes place on a transfinite cardinal number $\tau$ (and not necessarily on a transfinite number of the second number class).

**Lemma 2.** $E$ is compact in $\dot{L}$.

**Proof.** It is enough to verify the conditions of Arzelà' s theorem (Ascoli, see (2)).

**Lemma 3.** Let $\bar{R}_1 \subseteq R$ be invariant $(E)$. Then $\bar{R}_1$ is invariant $(E)$.

This is proved with the aid of Lemma 2.

**Theorem 1.** There exists a recurrent function $x(t) \in E$.

This is proved with the aid of Lemmas 1, 2, and 3.

Now suppose that $R$ is not invariant, but there exists $x_0(t)$–a solution of (1)–such that $x_0(t) \in R$ for $t \geq t_0$. Then we denote by $E_0$ the set of functions $x(t)$ –solutions of (1)–such that $x(t) \in R$ for $t \geq t_1(x(t))$. From $E_0$ we construct $E$ as before.

**Lemma 4.** There exists $x(t) \in E$ defined on the whole line.

**Proof.** From the sequence of shifts $y_n(t) = T_n x_0(t)$ one can choose a convergent subsequence (Lemma 2). The domain of definition of its limit is the entire line.

**Theorem 2.** Suppose there exists $x_0(t)$–a solution of (1)–such that $x_0(t) \in R$ for $t \geq t_0$. Then there is a recurrent function $x(t) \in E$, defined on the whole line.

**Proof.** With the aid of Lemmas 4, 3, and 1 we obtain the existence of a minimal $(E)$ set $\Sigma \subseteq R$. The rest is the same as in the proof of Theorem 1.

**Definition 3.** A solution $x_0(t)$ $(t \geq t_0)$ will be called **completely stable in the sense of Lyapunov** if, for every neighborhood of zero $U \subset L$, there exists a neighborhood of zero $V$ such that, for all $t'$, $t''$ for which $x(t') - x(t'') \in V$, one has $x(t'+t) - x(t''+t) \in U$ for all $t > 0$.

**Theorem 3.** Suppose the conditions of Theorem 2 are satisfied and suppose $x_0(t)$ is completely stable in the sense of Lyapunov. Then there is an almost periodic function $x(t) \in E$, defined on the whole line.

**Proof.** By Theorem 2 there is a recurrent function $x(t) \in E$. The proof of the almost periodicity of $x(t)$ is similar to the arguments in (1), p. 429.

In the case of unbounded $f(x,t)$, in order to avoid cumbersome formulations, let us consider the case when $L$ is normed. By the change of time

$$\tau(t) = \int_0^t a(\xi)\, d\xi,$$

where $a(t) = \max\{\|f(x,t)\|; 1\}$, the system (1) is reduced to the form $dx/dt = f_1(x,t)$, where $f_1(x,t)$ is already bounded. In the completion of this system there is a recurrent trajectory (Theorem 2). Under the inverse change of time (time is thereby "compressed"), either the trajectory remains recurrent in the ordinary sense, or an "infinite velocity" appears on it; in the latter case it is certainly meaningful to call it recurrent.

In conclusion I express my gratitude to V. V. Nemytskii for posing the problem and for his attention to the work.

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## REFERENCES

1. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow-Leningrad, 1949.

2. N. Bourbaki, *Topologie générale* (Fascicule de résultats), Paris, 1953, ch. X, § 4, Théorème 1.

*Note: Figure translations are in progress. See original paper for figures.*