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Abstract

Full Text

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ON THE CONTINUITY AND COMPLETE CONTINUITY OF P. S. URYSON OPERATORS

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Let the function $K(t,s,u)$ be defined for $t \in \Omega_2$, $s \in \Omega_1$, $-\infty < u < \infty$, where Ω_1 and Ω_2 are two sets of finite measure in Euclidean spaces. Consider the integral operator

$$Ax(t) = \int_{\Omega_1} K(t,s,x(s))\,ds. \tag{1}$$

The first subtle investigations of equations with such operators were carried out by P. S. Uryson ([1]). As is known (see, for example, ([2])), the study of these equations is substantially simplified if one succeeds in selecting functional spaces in which the operator (1) acts and possesses the properties of continuity, complete continuity, etc. Various criteria for continuity and complete continuity of the operator (1) in the space C were considered in the papers ([3–6]), in the spaces L_p—in the papers ([2], [7–14]), and in Orlicz spaces—in the papers ([15,16]).

In the present article new criteria are proposed for the continuity and complete continuity of the operator (1), acting from the space $L_p = L_p(\Omega_1)$ into the space $L_q = L_q(\Omega_2)$. As usual, by $L_p = L_p(\Omega)$ $(0 < p < \infty)$ we denote the space of all measurable functions defined on Ω for which

$$\|x\|_p = \left\{ \int_{\Omega} |x(t)|^p\,dt \right\}^{1/p} < \infty; \tag{2}$$

for $1 \le p < \infty$ the space L_p is a Banach space with norm (2), while for $0 < p < 1$ it is a complete metric space with metric $\rho(x,y) = (\|x-y\|_p)^p$. By $L_\infty = L_\infty(\Omega)$ we denote the Banach space of functions essentially bounded on Ω, with norm

$$\|x\|_\infty = \operatorname{vrai\,sup} |x(t)|. \tag{3}$$

It is assumed that the kernel $K(t,s,u)$ satisfies the Carathéodory conditions, i.e., that $K(t,s,u)$ is measurable in the aggregate of the variables $t,s \in \Omega_2 \times \Omega_1$ for all u, and is continuous in u for almost all $t,s \in \Omega_2 \times \Omega_1$.

1. In ([13]) the following general criterion for the continuity of the operator (1) in the spaces L_p is proved.

Theorem 1. Let the functions $K(t, s, u)$ and $R(t, s, u)$ satisfy the Carathéodory conditions, and suppose that

$$|K(t, s, u)| \leq R(t, s, u). \tag{4}$$

Let the Uryson operator with kernel $R(t, s, u)$ act from L_p into L_q $(0 < q < \infty)$ and be continuous.

Then the Uryson operator with kernel $K(t, s, u)$ also acts from L_p into L_q and is continuous.

Suppose that the kernel $K(t, s, u)$ is nonnegative. For linear integral operators with nonnegative kernels, continuity already follows from the fact that this operator maps L_p into L_q. For nonlinear operators this assertion is false—there exist nonnegative kernels $K(t, s, u)$ for which the corresponding operator (1) maps L_p into L_q and does not have the property of continuity.

Theorem 2. Let the Uryson operator with nonnegative kernel $K(t, s, u)$ map L_p into L_q, where $0 < q < \infty$.

Then, for every set $\mathfrak{M}$ of functions with uniformly absolutely continuous norms in L_p, the equality

$$\lim_{\operatorname{mes} D \to 0} \sup_{x \in \mathfrak{M}} \left\| \int_D K(t, s, x(s))\, ds \right\|_q = 0. \tag{5}$$

The main idea of the proof of this theorem is borrowed from ([8]). An Uryson operator A, mapping L_p into L_q, with kernel $K(t, s, u)$, will be called **regular** if

$$\int_{\Omega_1} |K(t, s, u(t, s))|\, ds \in L_q$$

for every measurable function $u(t, s)$ $(s \in \Omega_1,\ t \in \Omega_2)$ satisfying the condition

$$|u(t, s)| \leq u_0(s) \in L_p \qquad (t \in \Omega_2).$$

For linear integral operators this notion of regularity coincides with the generally accepted one (see ([14], [17])).

To prove regularity of operator (1), estimates of the form (4) are usually used: if, under the hypotheses of Theorem 1, the operator with kernel $R(t, s, u)$ is regular, then the operator with kernel $K(t, s, u)$ is also regular. Let us also note that an integral operator (1) mapping L_p into L_q is regular if its kernel $K(t, s, u)$ is a nonnegative function, monotone and even in u.

Theorem 3. Every regular Uryson operator mapping L_p into L_q, where $0 < p \leq \infty,\ 0 < q < \infty$, is continuous.

The assertion of this theorem is false if $q = \infty$. Let us observe that there exist discontinuous Uryson operators mapping L_p into L_q (even with nonnegative kernels) which do not have the property of regularity.

2. **Theorem 4.** Let A be a regular Uryson operator mapping L_p into L_q, where $0 < q < \infty$. Let

$$\lim_{\operatorname{mes} D \to 0,\ \|x\|_p \le 1} \sup \left\| \int_D K(t, s, x(s))\, ds \right\|_q = 0. \tag{6}$$

Then the operator A is completely continuous.

The proof of this theorem is, in its main part, close to the proof of Theorem 1 from ([8]).

Recall (see ([12])) that an operator T mapping L_p into L_q ($q < \infty$) is called **improving** if it transforms every norm-bounded set $\mathfrak{M}$ of functions from L_p into a set of functions $T\mathfrak{M}$ with uniformly absolutely continuous norms in the space L_q. In ([12]) necessary and sufficient conditions are given under which the nonlinear operator

$$fx(s) = f(s, x(s))$$

($f(s, u)$ is a function of two variables satisfying the Carathéodory conditions) is improving. These conditions are formulated in the form of simple upper estimates on the function $|f(s, u)|$. For operator (1) one can likewise indicate simple sufficient conditions on $K(t, s, u)$ under which this operator is improving.

If it is known that some operator acting from L_p into L_q ($q < \infty$) is improving, then, in order to prove its complete continuity, it is enough to establish that this operator is compact in measure (see ([9], [10])).

Theorem 5. Let A be a regular Uryson operator acting from L_p into L_q, where $0 < p \le \infty,\ 1 \le q < \infty$. Suppose the operator acting from L_p into L_1

$$hx(s) = h(s, x(s)), \tag{7}$$

where

$$h(s, u) = \int_{\Omega_2} |K(t, s, u)|\, dt$$

is improving.

Then the operator A is compact in measure (and, for $q = 1$, completely continuous).

For operators with nonnegative kernels, the condition that h is an improving operator is equivalent to condition (6) when $q = 1$.

3. The criteria for complete continuity given in item 2 are of a general character. We give one particular criterion.

Theorem 6. Suppose the function $K(t, s, u)$ satisfies the inequality

$$|K(t, s, u)| \leq \sum_{i=1}^{n} R_i(t, s) f_i(s, u), \tag{8}$$

where $f_i(s, u)$ is a function defining the nonlinear operator

$$f_i x(s) = f_i(s, x(s)),$$

acting from L_p into L_{r_i}, and $R_i(t, s)$ is a kernel defining a linear integral operator acting from L_{r_i} into L_q. Suppose $q < \infty$ and, for each i, one of the following conditions is fulfilled:

a) $r_i \geq 1$, the operator f_i is improving;

b) $r_i > 1$, the operator R_i is completely continuous.

Then the Uryson operator with kernel $K(t, s, u)$ acts from L_p into L_q and is completely continuous.

4. In studying the operator (1) with a given kernel, the spaces L_p, L_q are usually not prescribed. Therefore the following is of interest.

Theorem 7. Let the Uryson operator act from L_p into L_q $(0 < p, q < \infty)$ and be regular. Then A, as an operator from L_{p_1} into L_q, where $p_1 > p$, is completely continuous.

We note that A, as an operator from L_p into L_{q_1} with $q_1 < q$, need not be completely continuous (this is an essential difference between nonlinear integral operators and linear ones). It follows from Theorem 5 that a regular Uryson operator A acting from L_p into L_q $(1 < q < \infty)$ will be a completely continuous operator acting from L_p into L_{q_1}, where $q_1 < q$, if the operator (7) (acting from L_p into L_1) is improving.

5. Above we did not consider Uryson operators with values in the space L_∞. We give simple criteria for continuity and complete continuity of such operators. In the case of linear operators, the conditions of these theorems are not only sufficient but also necessary.

Theorem 8. Suppose the function $K(t, s, u)$ satisfies the conditions:

a) for every set $\mathfrak{M}$ of functions with uniformly absolutely continuous norms in L_p, the relation

$$\lim_{\text{mes}\, D \to 0} \sup_{x \in \mathfrak{M}} \left\| \int_D K(t, s, x(s))\, ds \right\|_\infty = 0;$$

b) for every R

$$\lim_{\delta\to 0}\left\|\int_{\Omega_1}\sup_{\substack{|u_1|,\ |u_2|\le R,\\ |u_1-u_2|\le\delta}}|K(t,s,u_1)-K(t,s,u_2)|\,ds\right\|_\infty=0.$$

Then the Uryson operator with kernel $K(t,s,u)$ acts from L_p to L_∞ and is continuous.

Condition a) of this theorem is, in particular, fulfilled if

$$|K(t,s,u)|\le\sum_{i=0}^{n}R_i(t,s)|u|^{\delta_i},$$

where $0=\delta_0<\delta_1<\dots<\delta_n\le p$, and $R_i(t,s)$ are functions for which

$$\psi_i(t)=\int_{\Omega_1}|R_i(t,s)|^{p/(p-\delta_i)}\,ds\in L_\infty\quad(i=0,1,\dots,n),$$

and

$$\lim_{\operatorname{mes}D\to 0}\left\|\int_D R_0(t,s)\,ds\right\|_\infty=0.$$

Theorem 9. Let the function $K(t,s,u)$ satisfy the following conditions:

a) for each R, for almost all $t\in\Omega_2$, the inequality

$$\int_{\Omega_1}\sup_{|u|\le R}|K(t,s,u)|\,ds<\infty;$$

holds;

b) for any $R>0$ and $\varepsilon>0$ there exists a partition of the set Ω_2 into parts $\Omega^{(0)},\Omega^{(1)},\dots,\Omega^{(l)}$ such that $\operatorname{mes}\Omega^{(0)}=0$, and for each $i=1,\dots,l$

$$\int_{\Omega_1}\sup_{|u|\le R}|K(t',s,u)-K(t'',s,u)|\,ds<\varepsilon\quad(t',t''\in\Omega_i);$$

c)

$$\lim_{\operatorname{mes}D\to 0}\sup_{\|x\|_p\le 1}\left\|\int_D K(t,s,x(s))\,ds\right\|_\infty=0.$$

Then the Uryson operator with kernel $K(t, s, u)$ acts from L_p to L_∞ and is completely continuous.

6. Theorems 1–9 are valid for operators acting in spaces of vector-functions.

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* As M. A. Krasnosel' skii and E. I. Pustylnik informed the author, the following corrections must be made to their paper (11): throughout the article the terms "continuous" or "completely continuous" operator should be understood as "bounded continuous" or "bounded compact" operator; in the definition of the regularizer $\Phi(t, s, u)$ and in the conditions of the lemma the word "bounded" was omitted.

Note: Figure translations are in progress. See original paper for figures.