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Abstract

Full Text

Mathematics

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On the Question of Necessary and Sufficient Conditions for an Absolute Minimum

(Presented by Academician L. S. Pontryagin on 2.III.1965)

We consider the problem of the absolute minimum of the functional

$$I(x,u)=\int_A f^0(t,x,u)\,dt+F(x(\tau)),\tag{1}$$

where $t=(t^1,\dots,t^m)$, $x=(x^1,\dots,x^n)$, $u=(u^1,\dots,u^r)$ are elements of the vector spaces T, X, and U, respectively, and A is a closed domain in the space T, bounded by a continuous, piecewise-smooth hypersurface S, with $t=\tau$ on S. Define the set E of all pairs of functions $x(t)$, $u(t)$, defined on the domain A, such that $x^i(t)$ $(i=1,\dots,n)$ are continuous and have bounded partial derivatives for $t\in A$, while $u^k(t)$ $(k=1,\dots,r)$ are continuous everywhere on A, except for a finite number of points, and at the points of discontinuity have bounded partial derivatives. Next, define the set $D\subset E$ of pairs of functions $x(t)$, $u(t)$ satisfying, in addition to the indicated conditions, the system of (nm) differential equations in partial derivatives

$$\frac{\partial x^i}{\partial t^j}=f^i_j(t,x,u)\quad(i=1,\dots,n;\ j=1,\dots,m).\tag{2}$$

The functions $f^i_j(t,x,u)$ and the function $f^0(t,x,u)$ are continuous together with all first-order partial derivatives for $t\in A$, $x,u\in E$. The functional $F(x(\tau))$ is defined on the set of values $x(\tau)$ of the function $x(t)$ on the surface S and is bounded. The set of admissible values $x(t)$ and the set of admissible controls $u(t)$, considered for fixed $t\in A$, may be closed sets in the corresponding spaces X and U.

The aggregate of points $t\in A$ and the corresponding values $x(t)$ at these points determines a certain set B in the $(n+m)$-dimensional space $T\times X$. Let $B(t)$ be the "section" of the set B by some fixed value $t\in A$. On the set B we shall consider continuous functions $\varphi_j(t,x)$ $(j=1,\dots,m)$ possessing continuous partial derivatives. Using Ostrogradsky' s formula and the relations (2), we have $(x,u\in D)$:

$$\int_A \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \frac{\partial \varphi_j(t,x)}{\partial x^i} f_j^i(t,x,u) + \frac{\partial \varphi_j(t,x)}{\partial t^j} \right] dt = \int_S \sum_{j=1}^{m} \varphi_j(\tau,x) \cos(n,t^j)\, d\tau, \tag{3}$$

where n is the direction of the outward normal to the surface S. If we now introduce into consideration the functions

$$R(t,x,u) = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \frac{\partial \varphi_j(t,x)}{\partial x^i} f_j^i(t,x,u) + \frac{\partial \varphi_j(t,x)}{\partial t^j} \right] - f^0(t,x,u), \tag{4}$$

$$G(x(\tau)) = F(x(\tau)) + \int_S \sum_{j=1}^{m} \varphi_j(\tau,x) \cos(n,t^j)\, d\tau, \tag{5}$$

then the functional (1) can, by virtue of (3), be represented in the form

$$I(x,u) = G(x(\tau)) - \int_A R(t,x,u)\, dt \qquad (x,u \in D). \tag{6}$$

The following generalized principle of optimality holds:

Theorem 1. Let there be a pair $\bar{x}(t), \bar{u}(t) \in D$. In order that this pair minimize the functional (1) on the set D, it is necessary and sufficient that there exist functions $\varphi_j(t,x)$ $(j = 1, \dots, m)$ such that:

1) for all $t \in A^* = A \setminus S$, except for a finite number of points,

$$R(t,\bar{x}(t),\bar{u}(t)) = \sup_{x,u \in E} R(t,x,u); \tag{7}$$

2) for all $\tau \in S$

$$G(\bar{x}(\tau)) = \inf_{x(\tau) \in B(\tau)} G(x(\tau)). \tag{8}$$

The sufficiency of the indicated conditions was considered by V. F. Krotov in ($^{1-3}$).

Necessity. Let $P = P(t_0)$ be a fixed point of the domain A^*, and let δ be a nonnegative small parameter. Define the domain $P_\delta \subset A^*$, bounded by a sphere with center at the point P and volume δ, and families of so-called needle variations (see, for example, (^{4})) of the functions $x(t)$ and $u(t)$:

$$x_\delta(t) = \begin{cases} \bar{x}(t), & \text{for } t \in A^* \setminus P_\delta, \\ \tilde{x}(t), & \text{for } t \in P_\delta, \end{cases} \qquad (\delta \geq 0), \tag{9}$$

$$x_0(t) = \bar{x}(t);$$

$$u_\delta(t) = \begin{cases} \bar{u}(t), & \text{for } t \in A^* \setminus P_\delta, \\ \tilde{u}(t), & \text{for } t \in P_\delta, \end{cases} \qquad (\delta \geq 0), \tag{10}$$

$$u_0(t) = \bar{u}(t),$$

where $\tilde{x}, \tilde{u}$ is an arbitrary fixed pair of values from the set E. It can be shown that the families (9), (10) are admissible in the sense of (^{4}). Considering, on the totality of values $t \in A,\ x, u \in E$, the $(n+m)$-dimensional vector-function

$$\Phi(t, x, u) \equiv f(t, x, u) - dx/dt,$$

we define on the set E the functional

$$\Psi(x, u) = I(x, u) - L\Phi(t, x, u), \tag{11}$$

where L is some linear functional in the space of continuous vector-functions. Using the result obtained in (^{4}), one may assert that for the functional (11) the condition

$$\Psi'_\delta(x_\delta, u_\delta)\big|_{\delta=0} \geq 0 \tag{12}$$

is satisfied. The functional $\Psi(x, u)$ can be represented in the form

$$\Psi(x, u) = \int_A f^0(t, x, u)\, dt + F(x(\tau)) - \int_A \left\{ \sum_{j=1}^{m} \sum_{i=1}^{n} \left[f^i_j(t, x, u) - \frac{\partial x^i}{\partial t^j} \right] \frac{\partial \varphi_j(t, x)}{\partial x^i} \right\} dt,$$

where $\varphi_j(t, x)$ $(j = 1, \dots, m)$ are functions continuous on B with continuous partial derivatives. Finding the increment

$$\Delta\Psi(x_\delta, u_\delta) = \Psi(x_\delta, u_\delta) - \Psi(\bar{x}, \bar{u})$$

and then computing $\Psi'_\delta(x_\delta, u_\delta)\big|_{\delta=0}$, it is not difficult to obtain, by virtue of (4) and (12), for all P in A^*, except for a finite number of points, the inequality

$$R(t_0, \bar{x}(t_0), \bar{u}(t_0)) \geq R(t_0, \tilde{x}(t_0), \tilde{u}(t_0)),$$

which holds for any pair $\tilde{x}, \tilde{u} \in E$, equality being attained only when $\tilde{x}(t) = \bar{x}(t)$ and $\tilde{u}(t) = \bar{u}(t)$. Since the choice of the point P in A^* is arbitrary, the inequality just written is equivalent to (7). Taking into account, further, that $\bar{x}, \bar{u} \in D$, together with relation (7) we shall have

$$R(t, \bar{x}(t), \bar{u}(t)) = \sup_{x,u \in D} R(t, x, u).$$

Then from (6) we obtain that for all $\tau \in S$ relation (8) holds.

For simplicity of reasoning we assumed that an absolute minimum in the class D exists. If this assumption is not made, the following more general result can be formulated.

Theorem 1*. Let there be a sequence of pairs $\{x_s(t), u_s(t)\} \subset D$. In order that this sequence be minimizing for the functional (1) on the set D, it is necessary and sufficient that there exist functions $\varphi_j(t, x)$ $(j = 1, \dots, m)$ such that:

1) for all $t \in A^*$, except for a finite number of points,

$$\lim_{s \to \infty} R(t, x_s(t), u_s(t)) = r(t), \qquad r(t) = \sup_{x,u \in E} R(t, x, u);$$

2) for all $\tau \in S$

$$\lim_{s \to \infty} G(x_s(\tau)) = g(\tau), \qquad g(\tau) = \inf_{x(\tau) \in B(\tau)} G(x(\tau)).$$

If the functional $F(x(\tau))$ contains one fixed $x = x^*(\tau)$ or, in other words, the functional being minimized has the form

$$I(x, u) = \int_A f^0(t, x, u)\, dt + F^*(\tau), \tag{13}$$

then from Theorem 1 it follows

Theorem 2. Let there be a pair $\bar{x}(t), \bar{u}(t) \in D$. In order that this pair minimize the functional (13) on the set D, it is necessary and sufficient that there exist functions $\varphi_j(t, x)$ $(j = 1, \dots, m)$ such that, for all $t \in A^*$, except for a finite number of points,

$$R(t, \bar{x}(t), \bar{u}(t)) = \sup_{x,u \in E} R(t, x, u).$$

For the case of one independent variable $(m = 1)$

$$I(x, u) = \int_0^{t_1} f^0(t, x, u)\, dt + F(x(0), x(t_1)), \tag{14}$$

and a consequence of Theorem 1 will be

Theorem 3. Let there be a pair $\bar{x}(t), \bar{u}(t) \in D$. In order that this pair minimize the functional (14) on the set D, it is necessary and sufficient that there exist a function $\varphi(t, x)$ such that:

1) for all $t \in (0, t_1)$, except for a finite number of points,

$$R(t, \bar{x}(t), \bar{u}(t)) = \sup_{x,u \in E} R(t, x, u); \tag{15}$$

2)

$$G(\bar{x}(0), \bar{x}(t_1)) = \inf_{x(0) \in B(0),\, x(t_1) \in B(t_1)} G(x(0), x(t_1)), \tag{16}$$

where

$$G(x(0), x(t_1)) = F(x(0), x(t_1)) + \varphi(t_1, x(t_1)) - \varphi(0, x(0)).$$

Remark 1. The sufficiency of conditions (15), (16) for the absolute minimum of the functional (14) was established in [1]. Moreover, Krotov in [1,2] put forward the intuitive conjecture that these conditions are at the same time necessary. The properties of the functions $f^i(t, x, u)$ $(i = 0, 1, \ldots, n)$ indicated there differ somewhat from those adopted in the present note and are, evidently, insufficient for the validity of the conjecture stated.

Remark 2. Along with the case considered of a finite number of points at which $u^k(t)$ cease to be continuous, one may assume the presence, in these functions, of singularities on a set of points of measure zero, on a finite number of lines, etc. In accordance with the assumptions indicated, condition (7) in Theorem 1 and the analogous conditions in the subsequent theorems must be satisfied for all $t \in A^*$, except for the corresponding set of points (lines).

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