SYSTEMS OF THE FIRST DEGREE OF NON-STRUCTURAL STABILITY ON THE TORUS

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SYSTEMS OF THE FIRST DEGREE OF NON-STRUCTURAL STABILITY ON THE TORUS

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Systems of the "first degree of non-structural stability," which are relatively structurally stable among the set of non-structurally stable systems, have been studied in detail on the plane and on the cylinder in works ([1–3]). It is natural to transfer the consideration of such dynamical systems to other surfaces as well. Whereas on an orientable surface of genus $p = 0$ (on the sphere) the situation is analogous to the planar case, on surfaces of genus $p \geq 1$ fundamental differences arise.

In the present paper we consider dynamical systems of the first degree of non-structural stability*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

defined on the torus ($p = 1$). What is new (in comparison with propositions completely analogous to the theory of systems of the first degree of non-structural stability on the plane) in the theory of systems of the first degree of non-structural stability on surfaces, in particular on the torus, are propositions on the absence in these systems of trajectories winding from a double limit cycle or from a separatrix issuing from a saddle to the same saddle, and winding again onto that same limit cycle or separatrix issuing from a saddle to the same saddle, as well as the absence of Poisson-stable nonclosed semitrajectories. Whereas the proof of the first proposition presents no difficulties, the proposition on the absence in systems of the first degree of non-structural stability of Poisson-stable nonclosed semitrajectories requires additional considerations. Below a proof of the latter proposition is given by using the "rotation of the field"

$$\dot{x} = P(x, y) - \mu Q(x, y), \qquad \dot{y} = Q(x, y) + \mu P(x, y). \tag{2}$$

Considering the torus unfolded onto the plane (x, y), we shall call points congruent if their coordinates coincide modulo 2π. As is known ([4]), the following proposition is valid:

Lemma 1. *Suppose that on the torus there is a Poisson-stable nonclosed semi-trajectory. Then, whatever* $\varepsilon > 0$ *is taken, it is always possible on the plane* (x, y) *to find a triple of trajectories* L_1, L, L_2*, corresponding on the torus to nonclosed trajectories Poisson-stable in both directions, such that* L_1 *is above* L*,* L_2 *is below* L*, and* L_1, L_2 *belong to the* ε*-neighborhood of* L*.*

Lemma 2. *Suppose that for sufficiently small* $\varepsilon > 0$ *on the plane* (x, y) *there exists a triple of trajectories* L_1, L, L_2 *of system (1), such that* L *corresponds on the torus to a nonclosed trajectory Poisson-stable in both directions;* L_1, L_2 *have values of the phase coordinates* $x(t), y(t)$ *tending to infinity as* $t \to \pm\infty$*;* L_1 *lies above* L*,* L_2 *below* L*;* L_1, L_2 *belong to the* ε*-neighborhood of the trajectory* L*, and between* L_1, L_2 *there are neither equilibrium states nor limit cycles of system (2).*

Then, whatever sufficiently small $\mu^* > 0$ *is taken, in the interval* $\mu \in (-\mu^*, \mu^*)$ *there will be found two countable sequences:* $\mu_1, \mu_2, \dots, \mu_r, \dots$ $(0 < \dots < \mu_r < \mu_{r-1} < \dots < \mu_1 < \mu^*)$*,* $\mu_1', \mu_2', \mu_r', \dots$ $(-\mu^* < \mu_1' <$

* The functions $P(x, y)$, $Q(x, y)$ are 2π-periodic in x, y, and belong to class C^n or to the analytic class.

$< \mu_2' < \dots < \mu_r' < \dots < 0)$, for which there exist nonhomologous to zero closed trajectories of system (2) with pairwise distinct rotation numbers ([5,6])*.

Proof. Without loss of generality we may assume that the trajectory L of system (1) has the form

$$x = \varphi(t), \qquad y = \psi(t), \qquad \varphi(0) = \psi(0) = 0 \tag{3}$$

and that the value $P(0,0) > 0$. Then for sufficiently small $\varepsilon > 0$, for all points $M(2\pi m, 2\pi n)$ belonging to the domain D (D is the domain between L_1 and L_2), the value $P(2\pi m, y) > 0$, where m, n are a subset of the numbers $0, \pm1, \pm2, \dots$, and y is any value between the ordinates of the points of intersection of the trajectories L_1, L_2 with the lines $x = 2\pi m$. Let y_1, y_2 be the ordinates of the points of intersection of the trajectories L_1, L_2 with the line $x = 0$. We have: $D = D_1 \cup D_2 \cup D_3 \cup D_4$, where D_1 is the domain lying above L to the right of $x = 0$, D_2 is above L to the left of $x = 0$, D_3 is below L to the left of $x = 0$, and D_4 is below L to the right of $x = 0$. Denote by L^+ and L^- the positive and negative semitrajectories distinguished from L and having their beginning at the point $O(0,0)$. On the axis $x = 0$ there must necessarily exist two countable sequences tending to $O(0,0)$ of points $G(0,y)$, $G'(0,y')$, where either $y \in (0, y_1)$, $y' \in (0, y_1)$, or $y \in (y_2, 0)$, $y' \in (y_2, 0)$, satisfying the following condition: the positive semitrajectories of system (1) issuing from $G(0,y)$ pass through the points $M(2\pi m, 2\pi n)$ belonging either to the domain D_1 when $y \in (0, y_1)$, or to the domain D_4 when $y \in (y_2, 0)$; the negative semitrajectories of system (1) issuing from $G'(0,y')$ pass through the points $M'(2\pi m', 2\pi n')$ belonging either to the domain D_2 when $y' \in (0, y_1)$, or to the domain D_3 when $y' \in (y_2, 0)$, with m, n, m', n' being subsets of the numbers $0, \pm 1, \pm 2, \ldots$, and $m \to +\infty$ as $y \to 0$, $m' \to -\infty$ as $y' \to 0$. Using the set $G(0, y)$, considered, for example, for $y \in (0, y_1)$, and assigning any sufficiently small value $\mu^* > 0$, we construct a sequence $\mu_1, \mu_2, \ldots, \mu_r, \ldots,\ 0 < \cdots < \mu_r < \mu_{r-1} < \cdots < \mu_1 < \mu^*$, for which there exist closed trajectories of system (2) with pairwise distinct rotation numbers.

For this purpose consider the trajectory L_μ of system (2)

$$x = \varphi(t, \mu), \qquad y = \psi(t, \mu), \qquad \varphi(0, \mu) = \psi(0, \mu) = 0, \qquad \mu \in [0, \mu^*]. \quad (4)$$

Two cases are possible: 1) the trajectory L_{μ^*} exits the domain D_1 for some value $x = 2\pi x^*$, $x^* > 0$; 2) the trajectory L_{μ^*} does not exit the domain D_1 for any value $x \in (0, +\infty)$. In both cases, for all $t > 0$ the trajectory L_{μ^*} is certainly above the trajectory L.

In the first case one can always choose a value $2\pi m_1 > 2\pi x^*$, $M_1(2\pi m_1, 2\pi n_1) \in D_1$, since, according to the choice of the set $G(0, y)$, in the domain D_1 there exists a countable set of points of the form $M(2\pi m, 2\pi n)$.

Denote by A, B, C, respectively, the points of intersection of L_{μ^*} with L_1, of L_1 with $x = 2\pi m_1$, and of L with $x = 2\pi m_1$. The arcs AB and BC are arcs without contacts for the trajectories L_μ of system (2) for any $\mu \in [0, \mu^*]$. Since for $\mu = 0$ L_μ passes through C, and for $\mu = \mu^*$ L_μ passes through A, there exists a value μ_1, $0 < \mu_1 < \mu^*$, for which L_{μ_1} closes on the torus, since it passes through $M_1(2\pi m_1, 2\pi n_1)$. We shall show that there exists a value μ_2, $0 < \mu_2 < \mu_1 < \mu^*$, such that the trajectory L_{μ_2} passes through the point $M_2(2\pi m_2, 2\pi n_2) \in D_1$, and moreover $n_1/m_1 \neq n_2/m_2$. Since L_{μ_1} passes through the points $O(0, 0)$ and $M_1(2\pi m_1, 2\pi n_1)$, it will also pass through the points $M(2\pi m_1 k, 2\pi n_1 k)$. By virtue of the choice of the set $G(0, y)$ and by the theorem on continuous dependence on initial conditions for system (1), there will be found a countable set of trajectories L', congruent to L and passing through points $M'(2\pi m', 2\pi n') \in D_1$, moreover such that they intersect the line CM_1

* From Lemma 2 there follows a proof, simpler than in (7), of the theorem on the absence, in Poisson-stable rough dynamical systems on the torus, of nonclosed trajectories.

between the points C and M_1, i.e., from the moment of reaching the line $x = 2\pi m_1$, remaining between the trajectories L and L_{μ_1}. Since the trajectories L' form a countable set, there exists a countable subset of trajectories $L'' \subset \{L'\}$ whose points $M''(2\pi m'', 2\pi n'')$, belonging to the domain D_1, lie to the right of the line $x = 2\pi m_1$ and between L_{μ_1}, L. Therefore $n''/m'' \neq n_1/m_1$. Fix one of the trajectories L'' and denote $M''(2\pi m'', 2\pi n'')$ by $M_2(2\pi m_2, 2\pi n_2)$, where

$n_2/m_2 \neq n_1/m_1$. Analogously to what was indicated above, one can show that there exists a value $\mu_2 \in (0, \mu_1)$ for which L_{μ_2} passes through $M_2(2\pi m_2, 2\pi n_2)$. In exactly the same way one can show the existence of a countable sequence

$$\mu_1,\ \mu_2, \ldots, \mu_r, \ldots, \quad 0 < \cdots < \mu_r < \mu_{r-1} < \cdots < \mu_1 < \mu^*,$$

such that the corresponding trajectories $L_{\mu_1}, L_{\mu_2}, \ldots, L_{\mu_r}, \ldots$ pass through the points $M_1(2\pi m_1, 2\pi n_1), M_2(2\pi m_2, 2\pi n_2), \ldots, M_r(2\pi m_r, 2\pi n_r), \ldots$, and none of the ratios

$$n_1/m_1,\ n_2/m_2, \ldots,\ n_r/m_r, \ldots$$

is equal to another.

The second case, when the trajectory L_μ does not leave the domain D_1 for any value $x \in (0, +\infty)$, can be reduced to the first case by small perturbations. Considering the interval $\mu \in (-\mu^*, 0)$ and the set $G'(0, y')$, $y' \in (0, y_1)$, we obtain a sequence

$$\mu_1',\ \mu_2', \ldots, \mu_r', \ldots, \quad -\mu^* < \mu_1' < \cdots < \mu_{r-1}' < \mu_r' < \cdots < 0,$$

satisfying the requirements of Lemma 2. The case when in the set of points $G(0, y)$, $G'(0, y')$ one has $y \in (y_2, 0)$, $y' \in (y_2, 0)$, is completely analogous to the case when $y \in (0, y_1)$, $y' \in (0, y_1)$. The lemma is proved.

Theorem 1. In a dynamical system of the first degree of non-roughness on the torus there exists only a finite number of equilibrium states and limit cycles, and no combination of two non-rough trajectories is possible: equilibrium states (saddle-nodes, compound foci of first order), double limit cycles, or separatrices going from saddle to saddle.

Theorem 2. In a dynamical system of the first degree of non-roughness on the torus there cannot be Poisson-stable nonclosed semitrajectories.

Proof. Suppose the contrary. Then system (1) has on the torus a Poisson-stable nonclosed semitrajectory. Therefore the possible closed trajectories or closed contours composed of equilibrium states and their separatrices are homologous to zero. Using Lemma 1 and Theorem 1, it is not difficult to verify that the hypotheses of Lemma 2 are satisfied. Choose a number $\mu^* > 0$ from the condition that all non-rough systems (2) in the interval $(-\mu^*, \mu^*) \ni \mu$ have a topological structure analogous to system (1)*. We shall show that the interval $(-\mu^*, \mu^*)$ contains not a single value μ for which system (2) is a rough dynamical system on the torus. Otherwise, according to the definition of roughness ([8]), there exists an interval $(\alpha, \beta) \ni \tilde{\mu}$ such that system (2), considered at any value $\mu \in (\alpha, \beta)$, $\mu \neq \tilde{\mu}$, has a decomposition of the torus into trajectories topologically equivalent to the decomposition at $\mu = \tilde{\mu}$. Moreover, at least one of the endpoints of the interval (α, β), which we denote by μ^{**}, lies inside the interval $(-\mu^*, \mu^*)$ (since for $\mu = 0$ system (2) is a non-rough system on the torus), corresponds to an obviously non-rough system (2), and, moreover, system (2) at $\mu = \mu^{**}$ has a Poisson-stable nonclosed semitrajectory.

Then, by virtue of Lemma 2, it is always possible to find a value of the parameter

$$\mu \in (\alpha, \beta) \cap (-\mu^*, \mu^*)$$

such that system (2) at $\mu = \bar{\mu}$ has on the torus a closed trajectory not homologous to zero, with rotation number certainly different from the rotation numbers of any of the limit cycles (if they exist) of system (2) at the value $\mu = \tilde{\mu}$. However, by the choice of the number $\mu^* > 0$, system (2) at

$$\mu = \bar{\mu} \in (-\mu^*, \mu^*)$$

must have the same topological structure as at $\mu = \tilde{\mu}$, which is impossible.

* This follows from the definition of systems of the first degree of non-roughness ([1]).

Therefore, whatever $\tilde{\mu} \in (-\mu^*, \mu^*)$ we take, system (2) for the value $\mu = \tilde{\mu}$ must be a non-rough dynamical system on the torus and, in accordance with the choice of the number $\mu^* > 0$, must have on it a Poisson-stable nonclosed semitrajectory. But then, by Lemma 2, one can choose a value $\mu^{**} \in (-\mu^*, \mu^*)$, $\mu^{**} \neq \tilde{\mu}$, such that system (2) for $\mu = \mu^{**}$ has on the torus a closed trajectory nonhomologous to zero, which is impossible. The theorem is proved*.

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* The assertion of Theorem 2 is not violated if one considers dynamical systems in the class of "polynomials" of the given system N (7).