THE STRUCTURE OF PERIODIC LINEAR GROUPS AND ALGEBRAIC GROUPS

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Abstract

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THE STRUCTURE OF PERIODIC LINEAR GROUPS AND ALGEBRAIC GROUPS

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The following principal results are obtained in the paper:

1) The structure of periodic linear groups over an arbitrary field is established, and some examples of infinite simple periodic linear groups are given.

2) The conjugacy problem for Sylow p-subgroups in algebraic linear groups is completely solved. It is shown that, with the aid of periodic groups and with minimal use of topological methods, certain structural theorems in the theory of algebraic groups can be proved. The latter results, incidentally, were one of the incentives for the investigations in item I.

3) The inverse problem in the theory of discrete subgroups in Lie groups is solved. Namely, the structure is determined of algebraic groups and Lie groups whose finite subgroups are nilpotent or solvable.

The proofs, apart from the basic results of the theory of algebraic linear groups ([1–3]), finite groups, and Lie groups, rely on the recent results of J. Thompson and W. Feit ([4]), D. Hertzig ([5]), J. Thompson ([6]), A. Borel and Harish-Chandra ([7]), and the author ([8–10]).

In what follows, p is an arbitrary prime number, and q is the characteristic of the field Q, which is assumed arbitrary unless otherwise specified. Only linear algebraic groups are considered.

I. Periodic linear groups over a field of characteristic zero were studied already by I. Schur ([11]). According to Schur' s well-known theorem, every periodic linear group in this case has an abelian normal divisor whose index is finite and depends only on the degree of the linear group. This theorem reduces, to a considerable extent, the question of the structure of periodic linear groups over a field of characteristic zero to finite groups. The most important general structural theorems in the theory of finite groups (we have in mind the theorems of Sylow, Hall, Schur–Zassenhaus) carry over quite simply to this case.

At the same time, for fields of positive characteristic, already at the initial stage of study it becomes clear that periodic linear groups in this case are arranged in a very complicated way. Thus, for example, one can construct certain series of simple infinite periodic linear groups, and for a rather broad class of fields every linear group turns out to be periodic. Let, for example, Σ_0 be a simple field of characteristic $q > 0$, and let Σ be an infinite algebraic extension of Σ_0. It is easy to see that the full linear group $GL_n(\Sigma)$ is periodic. If $SL_n(\Sigma)$ is the special linear group, then the group $PSL_n(\Sigma)$ is simple and isomorphic to the linear group $Ad(SL_n(\Sigma))$, where $f : x \to Adx$ is the adjoint representation of $SL_n(\Sigma)$. Nevertheless, the basic theorems of finite groups prove to be true for arbitrary periodic linear groups, as is established in this paragraph.

The validity of Hall' s theorem ([12]) for solvable periodic linear groups was established by us in ([13]), while the local finiteness of perio-

periodic linear groups has been proved independently in ([14], [15]); therefore the main attention here is devoted to the Sylow theorems and the Schur–Zassenhaus theorem. We note that the theorems indicated above are not true, generally speaking, for arbitrary locally finite groups (even solvable ones) (see ([16])).

Theorem 1. *In every periodic linear group the Sylow p-subgroups are conjugate.*

The **proof** of Theorem 1 splits into two cases: 1) $p \neq q$; 2) $p = q$.

The first case is the basic one, and its proof is technically more complicated than in the second case. At the same time, the idea of the proof is fully illustrated by case 2.

Let Γ be a periodic linear group over the field Q. Denote by S_1 and S_2 arbitrary Sylow q-subgroups of Γ. Let $\mathfrak{M}_{S_1} = \{gS_1g^{-1}\}$, $g \in \Gamma$. From $\mathfrak{M}_{S_1}$ choose a group S_1^* such that $S_1^* \cap S_2 = R_{12}$ contains a normal divisor H_1 of the groups S_1^* and S_2, whose Q-envelope $[H_1]_Q$ has maximal dimension l. In view of the fact that the dimension of the envelope is bounded, such a choice can always be made. We show that then always $H_1 \neq (e)$. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k_1}$ be a maximal system of linearly independent elements of S_1, and $\mu_1, \mu_2, \dots, \mu_{k_2}$ an analogous system from S_2. Consider the group

$$L_{k_1,k_2} = \{\varepsilon_1, \dots, \varepsilon_{k_1}; \mu_1, \dots, \mu_{k_2}\}.$$

Since Γ is a locally finite group, L_{k_1,k_2} is a finite group. Let S_1' and S_2' be Sylow q-subgroups of L_{k_1,k_2} containing, respectively, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k_1}$ and $\mu_1, \mu_2, \dots, \mu_{k_2}$. Then the groups S_1' and S_2', and consequently also their centers Z_1' and Z_2', are conjugate in L_{k_1,k_2}. In view of the choice of ε_i and μ_i, it follows that $Z_1' \subset S_1$, $Z_2' \subset S_2$, and Z_1' and Z_2' are central subgroups. If $hZ_1'h^{-1} = Z_2'$, then $hS_1h^{-1} \in \mathfrak{M}_{S_1}$, and $hS_1h^{-1} \cap S_2 \supset Z_2' \neq (e)$.

Since S_2 is a q-group, it is reducible to a special triangular form. We shall assume that S_2 has already been reduced to this form. Then H_1 also has a special triangular form. By the invariance of H_1, $[H_1]_Q$ is also invariant with respect to the inner automorphisms produced by elements of the groups S_1^* and S_2. It is obvious that $[H_1]_Q$ has scalar-triangular form; hence it follows that the set of q-elements in $[H_1]_Q$ forms a subgroup H_1^q, whose normalizer contains S_1^* and S_2, which entails the inclusion $H_1^q \subset R_{12}$. Thus one may assume that the q-part of $[H_1]_Q$ coincides with H_1. Consider the group $\Phi = \{S_1^*, S_2\}$, in which

H_1 is a normal divisor, while S_1^* and S_2 are Sylow q-subgroups. Denote by $\overline{\Phi}$, $\overline{H}_1$ the Zariski closures of Φ and H_1, respectively. By a well-known theorem of C. Chevalley, there exists a rational homomorphism $\overline{f}$ of the group $\overline{\Phi}$ with kernel $\overline{H}_1$. Let f be the restriction of $\overline{f}$ to Φ. It is easy to see that $\operatorname{Ker} f = H_1$. Indeed, $\overline{H}_1 \cap \Phi$ is invariant in Φ and unipotent; consequently, $\overline{H}_1 \cap \Phi \subset S_1^* \cap S_2$, whence $\overline{H}_1 \cap \Phi = H_1$. Let $f(\Phi) = \Phi'$. Since H_1 is an invariant q-subgroup in Φ, $S_1^*/H_1 = \Delta_1$ and $S_2/H_1 = \Delta_2$ are Sylow q-subgroups in Φ'. As above, one can show that in $\mathfrak{M}^*_{\Delta_1}$ there exists a group Δ_1' such that $\Delta_1' \cap \Delta_2$ contains a nontrivial normal divisor H_1', the inverse image of which Δ' is a normal divisor in Φ, with $\Delta' \supset H_1$ and $\Delta' \subset R_{12}$, which contradicts the maximality of H_1. The contradiction obtained shows that $S_1^* = S_2$.

Theorem 2. *Let G be a periodic linear group possessing a normal divisor H such that the orders in H and G/H are relatively prime. Then in G there is a subgroup D such that $G = D \cdot H$, $D \cap H = (e)$, and all subgroups D possessing these properties are conjugate in G.*

The proof of Theorem 2 rests on the results of ([4]) and naturally splits into two cases: 1) H has no elements of order q; 2) H has elements of order q. The ideas of the proof of 1) and 2) are essentially different. We shall only note that in case 2) the proof is based on the method applied by the author in ([10]).

Example. Let Σ_k be a finite field of characteristic 5, consisting of 5^{3^k} elements. All Σ_k are regarded as realized as subfields of some algebraically closed extension of the prime field with 5 elements. Let $G_k = PSL(2, \Sigma_k)$. Then the group

$$G = \bigcup_{k=1}^{\infty} G_k$$

is a simple locally finite linear group, whose Sylow 2-subgroup has order 4.

The example constructed gives a negative answer to M. Kargapolov' s question ([17]) as to whether a locally finite group with finite Sylow 2-subgroups must have a locally solvable normal divisor of finite index, and to Question 7.1 from ([18])*. We note that this example is minimal, for, by results of J. Thompson and W. Feit, a locally finite group with cyclic Sylow 2-subgroups is locally solvable.

The following interesting problem remains unsolved: to describe all infinite simple periodic linear groups. In particular, is every such group countable?

II. In ([9]) the problem of conjugacy of Sylow p-subgroups in algebraic groups was solved by us for the main case $p \neq q$. However, the question of conjugacy of Sylow subgroups for $p = q$ remained unclear. It turned out that the methods developed in ([9]) are insufficient for this case. Only by using the delicate results of C. Chevalley ([3]) on semisimple algebraic groups and the results of D. Hertzig ([5]) and J. Thompson ([6]) on algebraic and finite groups with a regular automorphism was it possible to solve the conjugacy problem completely. At the same time, the new ideas made it possible somewhat to simplify the proof also for the case $p \neq q$.

Theorem 3. *Let G be an algebraic group on which there acts a group of inner automorphisms U^*, induced by a unipotent subgroup U. Then U^* leaves invariant some maximal connected unipotent subgroup of G.*

Corollary. *Every unipotent subgroup of a connected algebraic group G belongs to some unipotent Borel subgroup.*

At the same time we note that not every diagonalizable subgroup of G belongs to a torus. Such examples may be extracted, for instance, from ([19]).

From Theorem 3 and arguments analogous to those applied in ([9]), it follows that

Theorem 4. *In an algebraic linear group, Sylow p-subgroups $(p = q)$ are conjugate.*

After the results just presented, the question naturally arises of solving the problem of conjugacy of Sylow p-subgroups for algebraically nonclosed fields. First of all, let us note the positive solution of the question for the case of the real field. Indeed, for Lie groups with a finite number of connected components this was proved in ([13]), and every real algebraic group is of this kind ([7], [20]).

Theorem 5. *In every real algebraic linear group, Sylow p-subgroups are conjugate.*

At the same time, for fields far from algebraically closed, the situation changes completely; namely, there exist algebraic linear groups, even solvable ones—for example, over the field of rational numbers—which have infinitely many pairwise nonconjugate Sylow p-subgroups. It follows from this that the results of ([7]) on periodic subgroups of groups of units do not extend to the case of the field of rational numbers.

In conclusion to this section, we note that the known results on the structure of solvable and nilpotent algebraic groups ([1], [10])

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* M. Kargapolov informed the author that an analogous example, in a letter to him, was constructed by O. Kegel.

can easily be derived from Theorem 2, if one takes into account only the fact that in every torus over an algebraically closed field the periodic part is everywhere dense. The latter requirement is the only topological condition used here.

III. The main direction in the theory of discrete subgroups of Lie groups is connected with the description of these subgroups in various classes of Lie groups. Apparently, little is known concerning the inverse problem: the determination of those classes of Lie groups whose discrete subgroups possess given properties, i.e., the study of how the properties of discrete subgroups influence the properties of the group as a whole. It turns out that even finite subgroups have a very substantial influence on the structure of the group as a whole. We give the most important results.

Theorem 6. *A compact Lie group is nilpotent (solvable) if and only if its finite subgroups are nilpotent (respectively solvable).*

The second part of Theorem 6 can be formulated in several other forms.

Theorem 6′. *A compact Lie group is solvable if and only if it contains no finite subgroups isomorphic to* $SL(2,5)$ *and* $PSL(2,5)$.

Remark. Lie groups are considered to be infinite. If finite groups are also admitted, then we obtain classes of finite groups all of whose subgroups are nilpotent or solvable.

We now consider the general case. In the notation for real forms of Lie groups we follow ([21], [22]).

Theorem 7. *Let* G *be a connected Lie group containing no finite subgroups isomorphic to* $SL(2,5)$ *and* $PSL(2,5)$. *Then either* G *is solvable, or* $G = S \cdot R$, *where* R *is the radical and* S *is a semisimple Lie group. The simple components of* S, *up to local isomorphism, can be only of the following five types:* A_2^1, I_1, B_1^2, ID_1.

Corollary. *The structure of a connected Lie group with solvable or nilpotent finite subgroups is described by Theorem 7.*

Theorem 8. *An algebraic linear group whose finite subgroups are nilpotent is solvable, but it is not always nilpotent for* $q = 0$ *and is nilpotent for* $q > 0$.

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Note: Figure translations are in progress. See original paper for figures.