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Abstract

Full Text

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ON COMMUTATIVE ALGEBRAS OF OPERATORS IN THE SPACE Π_k

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1. Let R be a commutative algebra of bounded linear operators in the Pontryagin space* Π_k. The algebra R is called **symmetric** if from $A \in R$ it follows that $A^* \in R$, where A^* is the operator in Π_k defined by the condition $(Ax, y) = (x, A^*y)$, and (x, y) is the indefinite scalar product in Π_k. Two algebras R, R' in the spaces Π_k and Π'_k are called **equivalent** if there exists a one-to-one linear mapping of Π_k onto Π'_k that preserves the scalar product and maps R onto R'. The aim of the present paper is to describe, up to equivalence, commutative symmetric algebras (c.s.a.) in Π_k. For $k = 1$ this problem was solved by the author in ([2]).
2. Let R be a c.s.a. in Π_k. On the basis of Corollary 2 of Theorem 1 in ([3]) (see also ([4])) there exists a k-dimensional nonnegative subspace $\mathfrak{P}$, invariant with respect to all $A \in R$, and in $\mathfrak{P}$ there exists a vector $x \neq 0$ which is a common eigenvector for all $A \in R$, i.e.

$$Ax = \lambda(A)x \quad \text{for all } A \in R, \tag{2,1}$$

where $A \to \lambda(A)$ is a homomorphism of the algebra A into the field C of complex numbers. Every homomorphism $\lambda \to \lambda(A)$ of the algebra R into C is called an **eigenfunctional** (e.f.) of the algebra R, if there exists a nonnegative vector $x \neq 0$ for which (2,1) holds; the set

$$S_\lambda = \{x : x \in \Pi_k,\ (A - \lambda(A)1)^l x = 0 \text{ for some } l = l(x) \text{ and all } A \in R\}$$

is then called the **root lineal** of the e.f. λ. In view of the preceding, an e.f. always exists; the corresponding root lineals are invariant with respect to all $A \in R$. An e.f. λ is called **real** if $\lambda(A^*) = \overline{\lambda(A)}$, and **nonreal** otherwise.

I. *The nonreal eigenfunctionals, if they exist, form a finite set* $\lambda_1, \mu_1, \dots, \lambda_\sigma, \mu_\sigma$, *where* $\mu_j(A) = \overline{\lambda_j(A^*)}$; *the corresponding root lineals* S_{λ_j}, S_{μ_j} *are finite-dimensional zero coskew-associated subspaces and*

$$S_{\lambda_j} + S_{\mu_j} \perp S_{\lambda_l} + S_{\mu_l} \quad \text{for } j \neq l.$$

The space

$$H = \sum_{j=1}^{\sigma} \oplus (S_{\lambda_j} + S_{\mu_j})$$

is called the **hyperbolic space** of the algebra R.

It is obvious that H is a finite-dimensional space of type Π_k, invariant with respect to all $A \in R$; therefore its orthogonal complement $H^\perp$ is also invariant with respect to all $A \in R$, $\Pi_k = H \oplus H^\perp$, and the restriction of R to $H^\perp$ no longer has nonreal e.f. Therefore in what follows we shall assume that R has no nonreal e.f. in Π_k. Let $\mathfrak{P}$ be a nonnegative k-dimensional subspace invariant with respect to all $A \in R$; $\lambda_1, \dots, \lambda_p$ are all the distinct (real) e.f. of R with eigenvectors in $\mathfrak{P}$. Put $\rho_j = \dim(S_{\lambda_j} \cap \mathfrak{P})$.

II. *The functionals $\lambda_1, \dots, \lambda_p$ and the numbers $\rho_1, \dots, \rho_p$ do not depend on the choice of the k-dimensional subspace $\mathfrak{P}$, invariant with respect to all $A \in R$;*

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* For the definition and basic properties of the spaces Π_k and of operators in them, see ([1]).

and every l.f. R coincides with one of these functionals λ_j, $j = 1, \dots, p$. Put

$$\mathfrak{D}_j = \{x : x \in \Pi_k,\ (A - \lambda_j(A)1)^{\rho_j} x = 0 \quad \text{for all } A \in R\}; \qquad (2,2)$$

$$\mathfrak{D} = \sum_{j=1}^{p} \oplus \mathfrak{D}_j. \qquad (2,3)$$

$$\mathfrak{M} = \mathfrak{D}^\perp, \qquad \mathfrak{N} = \mathfrak{D} \cap \mathfrak{M}; \qquad (2,4)$$

$\mathfrak{D}_j, \mathfrak{D}, \mathfrak{M}, \mathfrak{N}$ are closed subspaces, invariant with respect to all $A \in R$; $\mathfrak{N}$ is the null subspace; $\mathfrak{D}$ contains every k-dimensional nonnegative subspace invariant with respect to all $A \in R$. Hence it follows that $\mathfrak{M}$ is nonpositive; $\mathfrak{D}, \mathfrak{M}, \mathfrak{N}$ are called, respectively, the principal, fundamental, and fundamental null subspace of the algebra R.

Let $\mathfrak{N}'$ be a subspace skew-connected with $\mathfrak{N}$. Put $\mathfrak{H} = \mathfrak{M} \cap \mathfrak{N}'^\perp$, $\Pi = \mathfrak{D} \cap \mathfrak{N}'^\perp$. Then $\mathfrak{H}$ is negative, i.e., essentially a Hilbert space, while Π is positive, negative, or of type Π_k,

$$\mathfrak{M} = \mathfrak{N} \oplus \mathfrak{H}, \qquad \mathfrak{D} = \mathfrak{N} \oplus \Pi; \qquad (2,5)$$

$$\Pi_k = (\mathfrak{N} + \mathfrak{N}') \oplus \mathfrak{H} \oplus \Pi \qquad (2,6)$$

(see ([1]), Theorem 4.1). If in this case $\mathfrak{N} = (0)$, then one should take $\mathfrak{N}' = (0)$, $\mathfrak{H} = \mathfrak{M}$, $\Pi = \mathfrak{D}$, and (2,6) passes into $\Pi_k = \mathfrak{M} \oplus \mathfrak{D}$.

3. Put $\mathfrak{N}_j = \mathfrak{N} \cap S_{\lambda_j}$. Let $j = 1, \ldots, q$; $q \leq p$, be all those values of j for which $\mathfrak{N}_j \neq (0)$ (if there are no such values of j, then, by definition, $q = 0$ and $\mathfrak{N} = (0)$); then $\mathfrak{N} = \sum_{j=1}^{q} \cdot \mathfrak{N}_j$. In $\mathfrak{N}_j$ there exists a basis x_{jl}, $l = 1, \ldots, r_j$; $j = 1, \ldots, q$, such that the matrix of each operator $A \in R$ in $\mathfrak{N}_j$ is triangular with $\lambda_j(A)$ on the diagonal, i.e.,

$$Ax_{jl} = \sum_{s=1}^{l} \lambda_{jls}(A) x_{js}, \tag{3,1}$$

where $\lambda_{jll}(A) = \lambda_j(A)$. Let y_{jl}, $l = 1, \ldots, r_j$; $j = 1, \ldots, q$, be a basis in $\mathfrak{N}'$, biorthogonal to x_{jl}, so that $(x_{jl}, y_{j'l'}) = \delta_{jj'}\delta_{ll'}$. Applying to $A^* y_{jl}$ the relation (2,6), we conclude that

$$A^* y_{jl} = \sum_{\mu=1}^{q} \sum_{s=1}^{r_\mu} \alpha_{jl\mu s}(A) x_{\mu s} + \sum_{\mu=l}^{r_j} \overline{\lambda_{j\mu l}(A)}\, y_{j\mu} + h_{jl}(A) + \pi_{jl}(A), \tag{3,2}$$

where $\alpha_{jl\mu s}(A)$ are numerical functions on R, and $h_{jl}(A)$, $\pi_{jl}(A)$ are vector-functions on R with values in $\mathfrak{H}$ and Π, respectively. Further, applying to Ah, $h \in \mathfrak{H}$, the first relation (2,5) and using (3,2), we obtain:

$$Ah = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h_{jl}(A)) x_{jl} + A_1 h, \tag{3,3}$$

where A_1 is an operator in $\mathfrak{H}$.

III. The correspondence $A \to A_1$ is a symmetric homomorphism, continuous in the operator norm, of the algebra R onto the symmetric commutative algebra $R_1 = \{A_1 : A \in R\}$ of operators in the Hilbert space $\mathfrak{H}$, where A_1^* is the ordinary adjoint operator to A_1.

The algebra R_1, up to equivalence, does not depend on the choice of the subspace $\mathfrak{N}'$ skew-connected with $\mathfrak{N}$.

Similarly we find that for $\pi \in \Pi$

$$A\pi = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (\pi, \pi_{jl}(A)) x_{jl} + A_2 \pi, \tag{3,4}$$

where A_2 is an operator in Π.

IV. The correspondence $A \to A_2$ is a continuous, in the operator norm, symmetric homomorphism of the algebra R onto a commutative symmetric algebra

$$R_2 = \{A_2 : \ A \in R\}$$

of operators in Π, and this algebra, up to equivalence, does not depend on the choice of the subspace associated with $\mathfrak{N}$.

From (2,2) and (2,3) it follows that

$$\Pi = \sum_{j=1}^{p} \oplus \Pi^j, \tag{3,5}$$

where each Π^j is a subspace, positive, negative, or of type Π_k, invariant with respect to all $A_2 \in R_2$, and, moreover,

$$(A_2 - \lambda_j(A)1)^{\rho_j} = 0 \quad \text{on } \Pi^j \text{ for all } A \in R. \tag{3,6}$$

We shall call an algebra R in a Hilbert space or in a space of type Π_k **degenerate** if there exists a homomorphism $A \to \lambda(A)$ of the algebra R into C and a natural number ρ such that $(A - \lambda(A)1)^\rho = 0$ for all $A \in R$. From (3,6) it follows easily that R_2 is degenerate on each Π^j. It is also easy to see that a degenerate algebra in a Hilbert space is simply an algebra of operators of multiplication by a number. The structure of degenerate algebras in a space Π_k will be described below in § 4.

Formulas (3,2) with A^* instead of A, and (3,1), (3,3), (3,4), completely determine the operators $A \in R$. To obtain now a description of all possible c.s.a. in Π_k, it remains only to rewrite these formulas in the form

$$Ax_{jl} = \sum_{s=1}^{l} \lambda_{jls} x_{js}; \tag{3,7}$$

$$Ah = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h_{jl}) x_{jl} + A_1 h; \tag{3,8}$$

$$A\pi = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (\pi, \pi_{jl}) x_{jl} + A_2 \pi; \tag{3,9}$$

$$Ay_{jl} = \sum_{\mu=1}^{q} \sum_{s=1}^{r_\mu} \alpha^*_{jl\mu s} x_{\mu s} + \sum_{\mu=l}^{r_j} \overline{\lambda}^*_{j\mu l} y_{j\mu} + h^*_{jl} + \pi^*_{jl}, \tag{3,10}$$

where the systems

$$\xi = \{\lambda_{jls}, \alpha_{jl\mu s}, h_{jl}, \pi_{jl}, A_1, A_2\},$$

which determine the operators $A \in R$, range over a certain linear manifold Ξ, on which an involution

$$\xi \to \xi^* = \{\lambda^*_{jls}, \alpha^*_{jl\mu s}, h^*_{jl}, \pi^*_{jl}, A^*_1, A^*_2\}$$

is defined and which satisfies a system of axioms expressing the fact that the corresponding operators A range over a c.s.a. Every such linear manifold Ξ will be called a **determining manifold** associated with the decomposition (2,6) and with the algebras R_1, R_2.

Thus, the following holds.

Theorem 1. Every c.s.a. in Π_k is determined by a decomposition of the form (2,6), c.s.-algebras R_1, R_2 in $\mathfrak{H}$ and Π, and a determining manifold Ξ associated with them. Here $\mathfrak{N}, \mathfrak{N}'$ are associated null subspaces of dimension $k_0 \leq k$; $\mathfrak{H}$ is negative; Π is negative, positive of dimension $k - k_0$, or a space Π_{k-k_0}, when $k > k_0$;

$$\Pi = \sum_{j=1}^{p} \oplus \Pi^j,$$

where each Π^j is nondegenerate, invariant with respect to all $A_2 \in R_2$, narrowed-

$\cdots R_2$ on Π^j is a degenerate algebra, and R consists of all operators A in Π_k defined by formulas (3,7)–(3,10), where ξ ranges over Ξ; x_{jl} is a basis in $\mathfrak{N}$, and y_{jl} is a biorthogonal basis in $\mathfrak{N}'$.

Conversely, every decomposition of the form (2,6), a.c.a. R_1, R_2, and the defining manifold Ξ associated with them determine in this way a c.c. algebra in Π_k.

4. The preceding construction can be simplified if, leaving everything else unchanged, $\mathfrak{L}$ is replaced by any k-dimensional nonnegative subspace $\mathfrak{P}$, invariant with respect to all $A \in R$, which we shall again denote here by $\mathfrak{L}$. Then Π will be finite-dimensional positive and therefore the corresponding degenerate algebras in Π^j will be algebras of multiplication by a scalar. However, this modified construction will now depend on the choice of $\mathfrak{P}$.

This modified construction can be applied to a degenerate algebra R in Π_k. In this case $q \leqslant 1$, R_1, R_2 are algebras of multiplication by a scalar.

5. Suppose now that Π_k is separable and that R is separable in the operator norm and contains the identity operator. Then one can obtain a final realization of the restriction of the algebra R to the principal space $\mathfrak{N}$. Let $\overline{R}_1$ be the closure of R_1 in the operator norm; T the bicompact space of maximal ideals t of the algebra $\overline{R}_1$; $A(t)$ the value of the element $A_1 \in \overline{R}_1$ at the ideal t. As is known (see, for example, ([5]), § 41 and Supplement 11, or ([6]), Ch. I, § 7 and Ch. II), R_1, up to equivalence, is realized in the following way. There exist a Borel measure σ and at most a countable family of closed sets

$$F_1 = T \supset F_2 \supset F_3 \supset \cdots$$

such that

$$\mathfrak{L} = \int_T \mathfrak{L}(t)\, d\sigma,$$

where $\mathfrak{L}(t)$ is a subspace of l^2 consisting of all such $h = \{h_1, h_2, ...\} \in l^2$ that $h_k = h_{k+1} = \cdots = 0$ for $t \in F_k$. Here $\overline{R}_1$ is the set of all operators $A_1\{h(t)\} = \{A(t)h(t)\}$, $A(t) \in C(T)$, $\{h(t)\} \in \mathfrak{L}$, $\{A(t), A_1 \in \overline{R}_1\}$ is dense in $C(T)$, and

$$A_1^*\{h(t)\} = \{\overline{A(t)}h(t)\}.$$

We shall call this realization of $\overline{R}_1$ and R_1 canonical; by III, it does not, up to equivalence, depend on the choice of $\mathfrak{N}'$.

Cf. λ_j will be called special if there exists a point $t_j \in T$ such that $\lambda_j(A) = A(t_j)$ for all $A \in R$; t_j is called the corresponding special point of R. Put $K_j = \mathfrak{L}(t_j)$, if t_j is a special point and $\sigma_j = \sigma(\{t_j\}) > 0$, and $K_j = (0)$ otherwise. K_j is a Hilbert space; it is called a special space of R.

V. In the canonical realization of the algebra R_1

$$h_{jl}(A) = h_{jl}(A,t) = (A(t) - \lambda_j(A))\xi_{jl}(t) - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(A)}\, \xi_{j\mu}(t) \quad \text{for } t \neq t_j, \quad (4,1)$$

$$h_{jl}(A, t_j) = k_{jl}(A) \in K_j,$$

where $\xi_{jl}(t) \in \mathfrak{L}(t)$; $\xi_{jl}(t)$ is a σ-measurable function of t such that the right-hand side of (4,1) belongs to $\mathfrak{L}$ for all $A_1 \in R_1$.

Theorem 1 and Proposition V give a realization, up to equivalence, of the s.c. algebras, which we shall call their canonical model. The condition for the equivalence of two canonical models will be indicated in a subsequent communication.

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Note: Figure translations are in progress. See original paper for figures.