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1965

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Abstract

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ASYMPTOTICS OF THE SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR AN ELLIPTIC EQUATION IN THIN DOMAINS

(Presented by Academician I. N. Vekua, February 20, 1965)

In the work ([3]) we constructed the asymptotics of the solution of a boundary-value problem for a second-order elliptic equation with respect to a small parameter, where the small parameter enters into the geometry of the domain. The purpose of this note is to transfer the results obtained in ([3]) to the boundary-value problem for an elliptic equation of arbitrary order.

Let Q be a cylinder in n-dimensional space of height h. We denote the lateral surface of this cylinder by F. Suppose that the direction of the axis x_n coincides with h, where h is sufficiently small in comparison with the other dimensions of Q. In Q consider the problem

$$Lu \equiv \partial^{2m} u/\partial x_n^{2m} + \mathcal{L}u = 0; \tag{1}$$

$$\partial^m u/\partial x_n^m\big|_{x_n=0} = 0, \quad \partial^m u/\partial x_n^m\big|_{x_n=h} = P, \quad \partial^k u/\partial x_n^k\big|_{x_n=0,\, x_n=h} = 0, \tag{2}$$

$$k = m+1, \dots, 2m-1;$$

$$\partial^i u/\partial \nu^i\big|_F = 0, \quad i = 0, 1, \dots, m-1, \tag{3}$$

where $P(x_1, \dots, x_{n-1})$ is a given smooth function; ν is the normal to the lateral surface of the cylinder Q; L is an elliptic operator, and $\mathcal{L}$ is an elliptic differential expression of order $2m$ in the variables $x_1, \dots, x_{n-1}$.

Make the change of variables $x_n = th$. In the new variables the problem (1), (2), and (3) is written as follows:

$$L_1 u \equiv h^{-2m} \partial^{2m} u/\partial t^{2m} + \mathcal{L}u = 0; \tag{4}$$

$$\partial^m u/\partial t^m\big|_{t=0} = 0, \quad h^{-m}\partial^m u/\partial t^m\big|_{t=1} = P, \quad \partial^k u/\partial t^k\big|_{t=0,\, t=1} = 0, \tag{5}$$

$$k = m+1, \dots, 2m-1,$$

$$\partial^i u/\partial \nu^i\big|_F = 0, \quad i = 0, 1, \dots, m-1. \tag{6}$$

We shall seek an asymptotic representation of the solution with respect to h in the form $u = \widetilde{w}_N + \tilde{v}_N + z_N$, where the function $\widetilde{w}_N$ will be determined by the first iterative process, $\tilde{v}_N$ by the second, and z_N is the remainder term.

First iterative process. To the splitting of the operator L in (4) there corresponds a recurrent process which is obtained if the approximate solution of equation (4) is sought in the form

$$\widetilde{w}_N = h^{-m} w_{-m} + h^{-m+1} w_{-m+1} + \dots + h^{-1} w_{-1} + w_0 + \dots + h_N^{Nw}. \tag{7}$$

Substituting the expression for $\widetilde{w}_N$ from (7) into (4) and (5) and comparing terms with equal powers of h, we obtain

$$\partial^{2m} w_{-m+i}/\partial t^{2m} = 0, \quad \partial^k w_{-m+i}/\partial t^k\big|_{t=0,\ t=1} = 0,$$

$$i = 0, 1, \dots, 2m-1; \quad k = m, m+1, \dots, 2m-1; \tag{8}$$

$$\partial^{2m} w_m/\partial t^{2m} = -\mathcal{L} w_{-m}; \tag{9}$$

$$\partial^m w_m/\partial t^m\big|_{t=0} = 0, \quad \partial^m w_m/\partial t^m\big|_{t=1} = P, \quad \partial^k w_m/\partial t^k\big|_{t=0,\ t=1} = 0, \tag{10}$$

$$k = m+1, \dots, 2m-1;$$

$$\partial^{2m} w_{m+i}/\partial t^{2m} = -\mathcal{L} w_{-m+i}, \quad \partial^k w_{m+i}/\partial t^k\big|_{t=0,\ t=1} = 0, \tag{11}$$

$$i = 1, 2, \dots; \quad k = m, \dots, 2m-1.$$

From problem (8) we find that

$$w_{-m+i} = \sum_{j=0}^{m-1} t^j w_{-m+i,j}, \quad i = 0, 1, \dots, 2m-1; \tag{12}$$

here all the functions $w_{-m+i,j}$ are to be determined below.

Taking (12), for $i = 0$, into account, from (9) and (10) we obtain

$$\partial^{2m} w_m / \partial t^{2m} = -\sum_{j=0}^{m-1} t^j \mathcal{L} w_{-m,j}, \tag{13}$$

$$\partial^m w_m / \partial t^m \big|_{t=0} = 0, \quad \partial^m w_m / \partial t^m \big|_{t=1} = P, \quad \partial^k w_m / \partial t^k \big|_{t=0,\ t=1} = 0, \tag{14}$$

$$k = m+1, \dots, 2m-1.$$

Since the adjoint homogeneous problem corresponding to problem (13) and (14) has a nonzero solution of the form $z = \sum_{j=0}^{m-1} c_j t^j$, in order for it to have a solution it is necessary and sufficient that

$$\int_0^1 \left\{ \sum_{j=0}^{m-1} t^j \mathcal{L} w_{-m,j} \cdot \sum_{j=0}^{m-1} c_j t^j \right\} dt = (-1)^{m+1} c_{m-1} P.$$

Hence, expanding the integral and comparing the expressions with identical coefficients c_j, we obtain

$$\mathcal{L} w_{-m,0} + 2^{-1} \mathcal{L} w_{-m,1} + \dots + m^{-1} \mathcal{L} w_{-m,m-1} = 0,$$

$$2^{-1} \mathcal{L} w_{-m,0} + 3^{-1} \mathcal{L} w_{-m,1} + \dots + (m+1)^{-1} \mathcal{L} w_{-m,m-1} = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(m-1)^{-1} \mathcal{L} w_{-m,0} + m^{-1} \mathcal{L} w_{-m,1} + \dots + (2m-2)^{-1} \mathcal{L} w_{-m,m-1} = 0,$$

$$m^{-1} \mathcal{L} w_{-m,0} + (m+1)^{-1} \mathcal{L} w_{-m,1} + \dots$$

$$\dots + (2m-1)^{-1} \mathcal{L} w_{-m,m-1} = (-1)^{m+1} P.$$

From the fact that the determinant of this system is different from zero, we find

$$\mathcal{L} w_{-m,j} = f_j(P), \quad j = 0, 1, \dots, m-1, \tag{15}$$

where $f_j(P)$ are known functions of the given function P. Thus we find that the unknown functions $w_{-m,j}$ must satisfy equations (15). Therefore,

$$w_m = \sum_{j=0}^{m-1} t^j w_{m,j} + \sum_{k=2m}^{3m-1} \frac{(k-2m)!}{k!} t^k f_{k-2m}(P).$$

In an analogous manner, continuing the process, we find

$$\mathcal{L} w_{-m+i,j} = 0, \quad i = 1, 2, \dots, 4m-1; \quad j = 0, 1, \dots, m-1.$$

Consequently,

$$w_{-m+i} = \sum_{j=0}^{m-1} t^j w_{-m+i,j}, \quad i = 1, 2, \dots, 4m-1.$$

At the next step we obtain

$$\partial^{2m} w_{3m}/\partial t^{2m} = -\mathcal{L} w_m, \quad \partial^k w_{3m}/\partial t^k \big|_{t=0,\ t=1} = 0,$$

$$k = m, m+1, \dots, 2m-1.$$

From the solvability condition for the last problem we find

$$\mathcal{L} w_{m,j} = \bar{f}_j, \qquad j = 0, 1, \dots, m-1,$$

where $\bar{f}_j$ are known functions of functions determined earlier. Hence,

$$w_{3m} = \sum_{j=0}^{m-1} t^j w_{3m,j} + F_{3m},$$

where F_{3m} is a known function. Summarizing what has been presented, we may write

$$w_i = \sum_{j=0}^{m-1} t^j w_{i,j} + F_i, \qquad i = -m, -m+1, \dots,$$

where each time F_i is a known function, and w_{ij} is determined from the following problems:

$$\mathcal{L} w_{ij} = f_{ij}, \qquad \partial^k w_{ij}/\partial \nu^k \big|_F = \varphi_{ijk}, \tag{16}$$

$$i = -m,\ -m+1, \dots; \qquad j, k = 0, 1, \dots, m-1.$$

The conditions on φ_{ijk} under which the last problems are solvable will be formulated below.

Obviously, the functions w_i ($i = -m,\ -m+1, \dots$), generally speaking, do not satisfy the boundary condition (3). Therefore, to these functions we add boundary-layer functions v_i, so that the resulting sum $w_i + v_i$ satisfies all boundary conditions. These functions v_i are determined by a second iterative process.

Second iterative process. To carry out the second iterative process in a sufficiently small neighborhood of F, we introduce local coordinates (ρ, y), where ρ is the distance along the normal, and $y = y(t, y_1, \dots, y_{n-2})$ are the coordinates of a point on F. Writing equation (4) in the new variables and making in the resulting equation the change of variables $\rho = ht$, we obtain

$$L_2 u \equiv h^{-2m}\left(\partial^{2m} u/\partial t^{2m} + A\partial^{2m} u/\partial \tau^{2m}\right) + h^{-2m+1}(\dots) + \dots = 0. \tag{17}$$

We seek an approximate solution of equation (17) in the form

$$\tilde{v}_N = h^{-m} v_{-m} + h^{-m+1} v_{-m+1} + \dots + h^{-1} v_{-1} + v_0 + \dots + h^N v_N, \tag{18}$$

so that

$$\partial^k v_i/\partial t^k\big|_{t=0,\, t=1} = 0, \qquad \partial^j v_i/\partial \tau^j\big|_{t=0} = -\,\partial^j w_{i-j}/\partial \nu^j\big|_F, \tag{19}$$

$$i = -m,\ -m+1, \dots; \qquad j = 0, 1, \dots, m-1; \qquad k = m, m+1, \dots, 2m-1;$$

here it is assumed that if $i - j < -m$, then $w_{i-j} \equiv 0$.

Writing the expression for $\tilde{v}_N$ from (18) and (17), comparing the coefficients of equal powers of h, and taking into account condition (19) for determining v_i, we obtain the following problems:

$$\partial^{2m} v_{-m}/\partial t^{2m} + A\frac{\partial^{2m} v_{-m}}{\partial \tau^{2m}} = 0; \tag{20}$$

$$\partial^k v_{-m}/\partial t^k\big|_{t=0,\, t=1} = 0, \qquad v_{-m}\big|_{\tau=0} = -w_{-m}\big|_F, \qquad \partial^i v_{-m}/\partial \tau^i\big|_{\tau=0} = 0, \tag{21}$$

$$k = m, m+1, \dots, 2m-1; \qquad i = 1, 2, \dots, m-1;$$

$$\partial^{2m} v_i / \partial t^{2m} + A\left(\partial^{2m} v_i / \partial \tau^{2m}\right) = \mathcal{P}_i, \qquad i = -m+1,\, -m+2, \dots; \tag{22}$$

$$\partial^k v_i / \partial t^i \big|_{t=0,\, t=1} = 0, \qquad \partial^j v_i / \partial \tau^j \big|_{\tau=0} = -\,\partial^j w_{i-j} / \partial \nu^j \big|_F, \tag{23}$$

$$k = m, m+1, \dots, 2m-1; \qquad j = 0, 1, \dots, m-1.$$

Before proceeding to find the function v_i of boundary-layer type, let us consider the problem

$$\psi^{(2m)}(t) + A\lambda^{2m}\psi(t) = 0; \tag{24}$$

$$\psi^{(k)}(0) = 0, \qquad \psi^{(k)}(1) = 0, \qquad k = m, m+1, \dots, 2m-1, \tag{25}$$

where $A > 0$ and λ is a parameter.

Denote by $\bar{\lambda}_{k,i}$ those eigenvalues of this problem for which $\operatorname{Re}\lambda_{k,i} < 0$, and by $\bar{\psi}_k(t)$ the corresponding eigenfunctions.

Theorem 1. *The system of eigenfunctions $\{\bar{\psi}_k(t)\}$ constitutes an m-fold complete system, and the expansion theorem holds.*

We note that the adjoint homogeneous problem corresponding to all the problems (20) and (21), (22) and (23) has a nonzero solution of the form $z = c_0\tau^m + c_1\tau^{m+1} + \dots + c_{m-1}\tau^{2m-1}$. Taking this fact into account, in order that the problem (20) and (21) have a solution of boundary-layer type at $\tau = 0$, it is necessary and sufficient that

$$\int_0^1 w_{-m}\big|_F \, dt = 0,$$

i.e.

$$\varphi_{-m,0,0} + \frac{1}{2}\varphi_{-m,1,0} + \dots + \frac{1}{m}\varphi_{-m,m-1,0} = 0. \tag{26}$$

Consequently, we specify the boundary functions from problem (16) for $i = -m$ so that (26) is satisfied. Then, on the basis of Theorem 1, we assert that problem (20) and (21) has a unique solution representable in the form

$$v_{-m} = \sum_{k=1}^{\infty} \sum_{i=1}^{m} c_{k,i} e^{\bar{\lambda}_{ki}\tau} \bar{\psi}_k(t). \tag{27}$$

By the method of mathematical induction it is proved that all the remaining functions v_i are also functions of boundary-layer type. At each step, from the solvability condition for problem (22) and (23), we find conditions on the functions φ_{ijk} from the first iterative process. Multiplying all v_i by a smoothing function, and denoting the functions obtained again by v_i, we thus obtain for the solution of the stated problem the representation

$$u = \sum_{i=-m}^{N+2m} h^i (w_i + v_i) + z_N, \tag{28}$$

where z_N is the solution of the problem

$$L_1 z_N = g, \qquad \partial^k z_N / \partial t^k \big|_{t=0, t=1} = 0, \qquad \partial z_N^i / \partial \nu^i \big|_F = 0,$$

$$k = m, \dots, 2m-1; \qquad i = 0, 1, \dots, m-1,$$

where $g = h^{N+1} g_1$, with g_1 a known function. The estimate

$$h^{-2m} \|\partial^m z_N / \partial t^m\|^2 + \sum_{i=1}^{m-1} \|\partial^m z_N / \partial x_i^m\|^2 + \|z_N\|^2 \le c\|g\|^2,$$

is valid, where c does not depend on h, and the norm is understood in the sense of the L_2 metric.

Thus the following has been proved.

Theorem 2. *Let $P(x_1, \dots, x_{n-1})$ be a sufficiently smooth function. Then for the solution of problem (1), (2), (3) there holds the asymptotic representation (27), where z_N tends to zero as $h \to 0$ like h^{N+1} in the L_2 metric.*

Taking this opportunity, I express my gratitude to L. A. Lyusternik and M. I. Vishik for formulating the problem and discussing the results.

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Received
21 I 1965

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.