ON THE PROBLEM OF M-CONVEXITY

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Abstract

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The problem of M-convexity, which arose in the theory of partial differential equations, is the most complete and natural form of the problem of solvability of systems of inhomogeneous equations or, equivalently, the problem of describing the cohomology of the sheaf of germs of solutions of homogeneous systems of equations with constant coefficients. We shall formulate this problem somewhat differently than other authors ([6,2]), and describe some results related to it.

Let n be a natural number; $x = (x_1, \dots, x_n)$ are coordinates in the space R^n; P is the ring of polynomials with complex coefficients in the n differential operators $\partial/\partial x_1, \dots, \partial/\partial x_n$. A general system of linear differential equations with constant coefficients is written in the following form:

$$pu = w, \tag{1}$$

where p is a certain matrix of size $t \times s$, $t, s > 0$, with elements from P, and u and w are vector-functions with s and t components, respectively.

Let us note in particular systems of this kind that are connected with the operator of exterior differentiation. The operator d, when applied to a differential form ω of order k, $k = 0, \dots, n-1$, defined in R^n, can be written as the action of a certain differential operator d_k with constant coefficients on a vector-function whose components are the coefficients of the form ω.

If $n = 2m$, we can introduce n variables $\zeta_j = x_j + ix_{m+j}$ and $\zeta_j^* = x_j - ix_{m+j}$, $j = 1, \dots, m$. The operator d can be decomposed into the sum $d' + d''$, and each of the operators d' and d'' analogously generates a series of differential operators d'_k, d''_k, $k = 0, \dots, m-1$. In particular, if $p = d''_0$ and $w = 0$, then system (1) becomes the Cauchy–Riemann system.

Let us return to the general system (1). Let Ω be a domain in R^n. By the symbol $\Phi(\Omega)$ we shall denote any of the spaces $\mathcal{E}(\Omega)$, $\mathcal{E}^\beta(\Omega)$ of infinitely differentiable functions and the spaces $D'(\Omega)$, $(D^\beta(\Omega))'$ of generalized functions (see ([4,7])). The possibility of unlimited differentiation in these spaces gives rise in them to structures of P-modules. For any natural k, by $[\Phi(\Omega)]^k$ we denote the direct sum of k copies of the module $\Phi(\Omega)$. By $\Phi_p(\Omega)$ we denote the subspace in $[\Phi(\Omega)]^s$ formed by the solutions of system (1) with $w = 0$. The aim of the following considerations is, ultimately, the study of the space $\Phi_p(\Omega)$.

To every matrix p of size $t \times s$ with elements from P one can associate the P-mapping $p : P^s \to P^t$, consisting in multiplication of vectors from P^s by this matrix (P^k is the direct sum of the rings P). To such a matrix p there corresponds the finite P-module $M = P^s/p^*P^t$, where p^* is the transposed matrix. Conversely, to every finite P-module M one can asso-

put in correspondence a certain matrix p such that $M \approx P^s/p^*P^t$. For this it is enough to write a chain of syzygies (see (1)) of the module M, i.e. an exact sequence of the form

$$\cdots \xrightarrow{p_2^*} P^{t_1} \xrightarrow{p_1^*} P^t \xrightarrow{p^*} P^s \to M \to 0, \tag{2}$$

where p, p_1, p_2 are certain matrices of the corresponding sizes, formed from elements of P.

The correspondence thus established between matrices p and finite P-modules is not one-to-one: isomorphic modules may correspond to different matrices. It turns out that the properties of the system (1) depend essentially only on the module $M = P^s/p^*P^t$; namely, if $M = P^s/p^*P^t \approx P^\sigma/q^*P^\tau$, then the spaces $\Phi_p(\Omega)$ and $\Phi_q(\Omega)$ are naturally P-isomorphic. Considering these spaces up to P-isomorphism, we shall denote them by $\Phi_M(\Omega)$.

Let $U = \{U_\alpha\}$ be some open covering of the domain Ω. By ${}^\nu\Phi_M(U)$, $\nu = 0, 1, 2, \ldots$, we denote the space of cochains on U of order ν with coefficients in the spaces $\Phi_M(U_{\alpha_0} \cap \cdots \cap U_{\alpha_\nu})$, endowed with the topology of the space $\Pi\Phi_M(U_{\alpha_0} \cap \cdots \cap U_{\alpha_\nu})$. By $\Phi_M(U)$ we denote the complex

$$0 \to \Phi_M(\Omega) \to {}^0\Phi_M(U) \xrightarrow{\partial_0} {}^1\Phi_M(U) \xrightarrow{\partial_1} \cdots, \tag{3}$$

where ∂_ν are the usual coboundary operators, and by $H_\nu(\Phi_M(U)) = \ker \partial_\nu / \operatorname{Im} \partial_{\nu-1}$ the cohomology of this complex. The following theorem, analogous to a well-known theorem of Dolbeault, relates this cohomology to the solvability of system (1).

Theorem 1. Let M be some finite P-module; let U be a convex covering of the domain Ω. Then the cohomology $H^\nu(\Phi_M(U))$ coincides with the cohomology of the complex

$$0 \to \Phi_M(\Omega) \to [\Phi(\Omega)]^s \xrightarrow{p_0} [\Phi(\Omega)]^t \xrightarrow{p_1} \cdots \quad (p_0 = p), \tag{4}$$

i.e. there are isomorphisms

$$H^k(\Phi_M(U)) \approx \operatorname{Ext}_P^k(M, \Phi(\Omega)), \qquad k \geqslant 1.$$

Moreover, if some mapping $\partial_\nu,\ \nu \geqslant 0$, of the complex (3) is a homomorphism, then the corresponding mapping p_ν of the complex (4) is also a homomorphism, and conversely.

We note that, since U is a convex covering, the spaces $H^k(\Phi_M(U))$ are algebraically isomorphic to the cohomology spaces in the domain Ω of the sheaf of germs of solutions of system (1) with $w = 0$. In particular, if $p = d_0('')$, then we may put $p_i = d_i('')$, $i = 1, 2, \ldots$. In this case the sequence (3) becomes the complex of differential forms with coefficients from $\Phi(\Omega)$, and Theorem 1 becomes the de Rham theorem (Dolbeault).

Definition. We shall say that the space $\Phi(\Omega)$ is M-convex if

$$\mathrm{Ext}_P^k(M, \Phi(\Omega)) = 0$$

for all $k \geqslant 1$, and the operators p_i in (4) are homomorphisms. We shall say that the space $\Phi(\Omega)$ is **strongly M-convex** if it is M-convex and the exponential polynomials belonging to $\Phi_M(\Omega)$ are dense in this space.

It can be shown that M-convexity and strong M-convexity depend only on the module M and do not depend on the choice of the sequence (2).

Since the space $\mathcal{E}(\Omega)$ is a Fréchet space, for its M-convexity it is enough that the relations

$$\mathrm{Ext}_P^k(M, \mathcal{E}(\Omega)) = 0, \qquad k \geqslant 1$$

hold. Therefore, by the Cartan–Oka–Serre theorem ([3]), the convexity of the space $\mathcal{E}(\Omega)$ with respect to the module $P/(d_0'')^* P^m$ $(n = 2m)$ is equivalent to the fact that Ω is a domain of holomorphy. The space $\mathcal{E}(\Omega)$ is strongly convex with respect to this module if and only if the domain Ω is a Runge domain.

Let us describe some properties of domains Ω such that the spaces $\Phi(\Omega)$ are M-convex. Let M be some finite P-module. By $\operatorname{ann} M$ we denote the ideal in P formed by all polynomials f for which $fF = 0$ for every element $F \in M$. By $\dim M$ we denote the dimension of the module M as a linear space over the field of quotients of the ring P.

Theorem 2. *If the ideal* $\operatorname{ann} M$ *is Noetherian, then in every domain* Ω *there are isomorphisms*

$$\mathrm{Ext}_P^k(M, \Phi(\Omega)) \simeq [H^k(\Omega, C)]^{\dim M}, \qquad k \geqslant 1.$$

These isomorphisms are topological if the space $H^k(\Omega, C)$ *is endowed with the topology of compact convergence.*

Theorem 3. *Let the dimension of the ideal* $\operatorname{ann} M$ *be equal to* d*. Then, if the space* $\Phi(\Omega)$ *is* M*-convex,* $H^k(\Omega, C) = 0$, $k > d$*. If the space* $\Phi(\Omega)$ *is strongly* M*-convex, then also* $H^d(\Omega, C) = 0$.

This theorem generalizes Serre' s theorem ([10]), which states that in any Runge domain Ω, $H^k(\Omega, C) = 0$ for $k \geqslant m$. Indeed, if Ω is a Runge domain, then the space $\mathcal{E}(\Omega)$ is strongly M-convex with respect to the module $M = P/(d_0'')^* P^m$, and the dimension of this module, as is not difficult to compute, is equal to m.

It is known ([11–13]) that in order that the space $\Phi(\Omega)$ be (strongly) M-convex with respect to any finite P-module M, it is necessary and sufficient that the domain Ω be convex. For each particular module M, the class of M-convex and strongly M-convex spaces is much broader. In the case $s = t$, the broadest class of strongly convex spaces is apparently possessed by elliptic modules, i.e. modules of the form $P^s/p^* P^s$, where p^* is an elliptic operator. This is supported by the following fact, established essentially by Malgrange ([5]): if $H^{n-1}(\Omega, C) = 0$, then the space $\mathcal{E}(\Omega)$ is strongly convex with respect to any elliptic module, and moreover this property of elliptic modules is characteristic. In connection with this, the following problem arises. Let an integer d, $1 \leq d < n$, be given. It is required to describe the class of modules M (i.e. differential operators with constant coefficients) possessing the property that the space $\Phi(\Omega)$ is strongly M-convex for any domain Ω such that $H^k(\Omega, C) = 0$, $d \leq k < n$.

To every finite module M we assign the sequence of finite modules

$$E_1(M) = \ker\{M \xrightarrow{j} \mathrm{Hom}_P(\mathrm{Hom}_P(M, P), P)\}, \qquad E_2(M) = \mathrm{Coker}\, j,$$

$$E_k(M) = \mathrm{Ext}_P^{k-2}(\mathrm{Hom}_P(M, P), P), \qquad k \geqslant 3,$$

where j is the canonical mapping. This sequence is finite, since, as always, $\mathrm{Ext}_P^k(M, P) = 0$ if $k > n$. The properties of these modules to a considerable extent determine the breadth of the class of M-convex spaces.

Let G be some closed set in R^n. By $\Phi(G)$ we denote the inductive limit of the spaces $\Phi(\Omega)$ over all neighborhoods Ω of the set G. The definition of M-convexity is extended in the obvious way to spaces of this kind.

We shall say that the set G is a **normal simplicial complex of dimension** k if two conditions are satisfied:

1. G is a finite union of sets G_λ homeomorphic to k-dimensional simplices in R^n, and any two simplices G_λ and G_μ either intersect in their $(k-1)$-dimensional face, or have no points in common.
2. There exists a k-dimensional subspace in R^n such that each simplex G_λ is projected one-to-one onto this subspace.

Theorem 4. *Let $E_i(M) = 0$, $i = 1, \dots, k$. Then for any normal simplicial complex G of dimension k, the space $\Phi(G)$ is M-convex.*

In particular, if $p = d_v$, then, as is not hard to calculate, $E_i(M) = 0$, $i \leq v - 1$. Consequently, the space $\Phi(G)$ is convex relative to the module corresponding to d_i, if $i \geq k + 1$. Let us note that this space need not be convex relative to the module corresponding to d_k, since it is not hard to construct a complex G of dimension k whose k-dimensional Betti number is different from zero.

Another geometric condition of convexity relative to modules is based on the notion of a q-convex domain. We construct this notion by analogy with the well-known notion of a q-pseudoconvex domain [8].

Let Ω be a domain in R^n. A function φ, defined in Ω and having continuous second derivatives in this domain, will be called q-convex, $0 \leq q \leq n$, if at every point $x \in \Omega$ the matrix $\{\partial^2\varphi/\partial x_i \partial x_j\}$ has at least q positive eigenvalues. We shall say that the domain Ω is q-convex if in Ω there exists a continuous function φ, q-convex outside some compact set $K \subset \Omega$, and such that every set $\{x : \varphi(x) \leq C < \infty\}$ is a compact set belonging to Ω. We shall say that the domain Ω is completely q-convex if, in addition, $K = \emptyset$. It is not hard to show that every convex domain is completely n-convex, and every bounded domain with smooth boundary is 1-convex.

Theorem 5. Let M be a hypoelliptic module [i.e. let $M \simeq P^s/p^*P^t$, where p is a hypoelliptic operator]. Then:

I. If the domain Ω is completely $(n-k)$-convex, then

$$\mathrm{Ext}^i_P(M, \mathcal{E}(\Omega)) = 0, \qquad i > k.$$

II. If the domain Ω is $n-k$-convex, then the spaces

$$\mathrm{Ext}^i_P(M, \mathcal{E}(\Omega)), \qquad i > k,$$

are finite-dimensional.

In particular, for any bounded domain Ω with smooth boundary the space $\mathrm{Ext}^{n-1}_P(M, \mathcal{E}(\Omega))$ is finite-dimensional. Let us consider another example: $p = d_0$. This operator is, obviously, hypoelliptic; therefore we arrive at the following result: if the domain Ω is $(n-k)$-convex, then the cohomologies $H^i(\Omega, C)$, $i > k$, are finite-dimensional; if, however, the domain Ω is completely $(n-k)$-convex, then these cohomologies are trivial. This fact is the real analogue of the Andreotti-Grauert theorem [9], which concerns q-pseudoconvex complex analytic spaces and the cohomologies of sheaves of germs of holomorphic functions on these spaces.

Let $n = 2m$. Consider operators of the form $p(D_{\xi^*})$, i.e. operators containing differentiations only with respect to the variables ξ^*. It turns out that the role of convex domains for such operators is played by holomorphically convex domains.

Theorem 6. In order that the space $\Phi(\Omega)$ be (strongly) M-convex relative to any module of the form $M = P^s/p^*(D_{\xi^*})P^t$, it is necessary and sufficient that the domain Ω be holomorphically convex (a Runge domain).

In the case $\Phi(\Omega) = \mathcal{E}(\Omega)$, this result is due to Malgrange [6].

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Note: Figure translations are in progress. See original paper for figures.