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Abstract

Full Text

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ON CHARACTERISTICS OF THE GROWTH OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

(Presented by Academician S. N. Bernstein on 25 V 1965)

For simplicity we shall restrict ourselves to the case of two complex variables; the case of an arbitrary finite number of variables is treated in the same way.

In studying the growth of entire functions of two complex variables, the space C^2 is usually exhausted either by the one-parameter family of surfaces $S(R) : |z_1|^\alpha + |z_2|^\alpha = R^\alpha$, $\alpha > 0$, or by the two-parameter family $S(r_1, r_2) : \{|z_1| = r_1\} \times \{|z_2| = r_2\}$. In the first case, the growth of a function $f(z_1, z_2)$ is measured by means of the function $\ln M(R)$, where $M(R)$ denotes the maximum modulus of the function $f(z_1, z_2)$ on $S(R)$, and one obtains global characteristics of growth —order and type. In the second case one compares the growth of the function $\ln M(r_1, r_2)$ with the growth of $\sigma_1 r_1^{\rho_1} + \sigma_2 r_2^{\rho_2}$, where

$$M(r_1, r_2) = \max_{(z_1, z_2) \in S(r_1, r_2)} |f(z_1, z_2)|,$$

and arrives at the definition of a system of conjugate orders (s.c.o.) and a system of conjugate types (s.c.t.) ⁽¹⁾.

For a more complete characterization of the growth of an entire function of one complex variable the notion of refined order is used ⁽²⁾. It carries over automatically to the case of functions of two variables if their growth is measured by means of $\ln M(R)$. Less trivial is the problem of studying refined orders when the space C^2 is exhausted by the family of surfaces $S(r_1, r_2)$.

Let an entire function $f(z_1, z_2)$ of finite order ρ have s.c.o. (ρ_1, ρ_2) . As a system of conjugate refined orders (s.c.r.o.) it is natural to choose systems of functions $(\rho_1(r_1, r_2), \rho_2(r_1, r_2))$ possessing certain properties. We propose the following

Definition 1. A system of nonnegative continuous functions $(\rho_1(r_1), \rho_2(r_2))$, defined for $0 \leq r_i < \infty$, $i = 1, 2$, will be called an s.c.r.o. if: 1) $\rho_i(r_i)$ is a differentiable function of the variable r_i , except, perhaps, for isolated points at which one-sided derivatives exist; 2)

$$\lim_{r_i \rightarrow \infty} \rho_i(r_i) = \rho_i < \infty;$$

3)

$$\lim_{r_i \rightarrow \infty} r_i \rho_i'(r_i) \ln r_i = 0 \quad (i = 1, 2).$$

Definition 2. Let $(\rho_1(r_1), \rho_2(r_2))$ be an s.c.r.o. If for the entire function $f(z_1, z_2)$ there exists a system of positive numbers (σ_1, σ_2) that satisfies the condition

$$\lim_{r_1+r_2 \rightarrow \infty} \frac{\ln M(r_1, r_2)}{\sigma_1 r_1^{\rho_1(r_1)} + \sigma_2 r_2^{\rho_2(r_2)}} = 1,$$

then the system of functions $(\rho_1(r_1), \rho_2(r_2))$ is called an s.c.r.o. of the function $f(z_1, z_2)$, and the system (σ_1, σ_2) is called the system of conjugate types (s.c.t.) corresponding to the s.c.r.o. $(\rho_1(r_1), \rho_2(r_2))$.

The existence of an s.c.r.o. for any entire function $f(z_1, z_2)$ follows from the following theorem, which generalizes Theorem 16 of Ch. I in ⁽²⁾.

Theorem 1. Let the entire function $f(z_1, z_2)$ have s.c.o. (ρ_1, ρ_2) ($\rho_i < \infty$, $i = 1, 2$). Then there exists an s.c.r.o. $(\rho_1(r_1), \rho_2(r_2))$ of the function $f(z_1, z_2)$, which satisfies the conditions: 4)

$$\ln M(r_1, r_2) \leq r_1^{\rho_1(r_1)} + r_2^{\rho_2(r_2)}$$

for $r_1 + r_2 > R_0$; 5) on some sequence $\{(r_1^{(k)}, r_2^{(k)})\}$

$$(r_1^{(k)} + r_2^{(k)}) \rightarrow \infty$$

the equality

$$\ln M(r_1^{(k)}, r_2^{(k)}) = (r_1^{(k)})^{\rho_1(r_1^{(k)})} + (r_2^{(k)})^{\rho_2(r_2^{(k)})}$$

holds.

On the other hand, it can be shown that for any s.s.r.o. $(\rho_1(r_1), \rho_2(r_2))$ there exists an entire function $f(z_1, z_2)$ for which $(\rho_1(r_1), \rho_2(r_2))$ is an s.s.r.o. In this case it is necessary to require that when $\rho_i(r_i) \rightarrow 0$, $\rho_i(r_i) \neq 0$, $i = 1, 2$, for at least one of the values $i = 1, 2$ condition 6) hold:

$$\lim_{r_i \rightarrow \infty} r_i^{\rho_i(r_i)} (\ln r_i)^{-1} > 0.$$

Obviously, the s.s.r.o. $(\rho_1(r_1), \rho_2(r_2))$ of the function $f(z_1, z_2)$ is not determined uniquely. In some cases it may be more convenient to use another definition of s.s.r.o., which is obtained if in definitions 1 and 2 the system of functions $(\rho_1(r_1), \rho_2(r_2))$ is replaced by the system $(\rho_1(R), \rho_2(R))$, where $\rho_i(R) = \rho_i(r_1 + r_2)$, as functions of the variable R , possess properties 1)–3). For an s.s.r.o. $(\rho_1(R), \rho_2(R))$ Theorem 1 also holds under one additional condition. Namely,

it is required that for an entire function $f(z_1, z_2)$ having s.r.o. (ρ_1, ρ_2) and not depending on the variable $z_1(z_2)$, the system (ρ_1, ρ'_2) with $\rho'_2 < \rho_2$ ((ρ'_1, ρ_2) with $\rho'_1 < \rho_1$) not be an s.r.o.

In what follows, when considering the s.s.r.o. $(\rho_1(r_1), \rho_2(r_2))$ ($(\rho_1(R), \rho_2(R))$), we assume that $\rho_i(r_i) \rightarrow \rho_i > 0$ ($\rho_i(R) \rightarrow \rho_i > 0$), $i = 1, 2$, without mentioning this specially.

As is known (see (2)), the function $t_i^{\rho_i(r_i)}$ increases monotonically for all sufficiently large values of r_i . Denote the inverse function by $\chi_i(t_i)$ ($t_i > t_i^0$) and extend it to the interval $[0, t_i^0]$, setting it equal to a certain positive constant. The following analogue of Theorem 2, Ch. I from (2), establishes a connection between s.s.r.o., s.t., and the coefficients of the expansion

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl} z_1^k z_2^l. \quad (1)$$

Theorem 2. *In order that the positive numbers σ_1, σ_2 constitute an s.t. for an s.s.r.o. $(\rho_1(r_1), \rho_2(r_2))$ of the function $f(z_1, z_2)$, it is necessary and sufficient that*

$$\lim_{k+l \rightarrow \infty} \sqrt[k+l]{|a_{kl}| \chi_1(k)^k \chi_2(l)^l (e\sigma_1 \rho_1)^{-k/\rho_1} (e\sigma_2 \rho_2)^{-l/\rho_2}} = 1. \quad (2)$$

For $\rho_i(r_i) \equiv \rho_i$, $i = 1, 2$, this theorem implies the known relation for s.s.r.o. and s.t., established by L. I. Ronkin ((1), p. 390).

An analogous theorem holds when considering the s.s.r.o. $(\rho_1(R), \rho_2(R))$. The role of the functions $\chi_i(t_i)$ in this case is played by the functions $r_i = \chi_i(t_1, t_2)$, $i = 1, 2$, which are the unique solution of the system of equations ($r_1 + r_2 \geq R_0$)

$$t_1 = r_1^{\rho_1(r_1+r_2)}, \quad t_2 = r_2^{\rho_2(r_1+r_2)}.$$

In this case equality (2) is replaced by the following:

$$\lim_{k+l \rightarrow \infty} \sqrt[k+l]{|a_{kl}| \chi_1(k, l)^k \chi_2(k, l)^l (e\sigma_1 \rho_1)^{-k/\rho_1} (e\sigma_2 \rho_2)^{-l/\rho_2}} = 1.$$

Hence the following assertion follows: whatever the s.s.r.o. $(\rho_1(R), \rho_2(R))$ may be, there exists an entire function $f(z_1, z_2)$ for which this system is an s.s.r.o. It can be shown that this assertion is also true in the case when only one of the functions $\rho_1(R), \rho_2(R)$ tends to zero as $R \rightarrow \infty$, or both tend to zero, but condition 6) is then satisfied for them.

Using Theorem 2, one can prove the following theorem, analogous to Theorem 26.4 from (1).

Theorem 3. *In order that a system of positive numbers (σ_1, σ_2) be an s.c.t. for an s.c.r.o. $(\rho_1(r_1), \rho_2(r_2))$ of some entire function $f(z_1, z_2)$, it is necessary and*

sufficient that the curve with coordinates $(\ln \sigma_1, \ln \sigma_2)$ constitute the boundary of some convex quadrant-like* domain D .

An analogous theorem is valid for an s.c.r.o. $(\rho_1(R), \rho_2(R))$.

Following V. K. Ivanov ^(3,4), we give the following

Definition 3. Let $f(z_1, z_2)$ be an entire function with s.c.r.o. $(\rho_1(r_1), \rho_2(r_2))$. By $T(\varphi) = T(\varphi_1, \varphi_2)$, $0 \leq \varphi_1, \varphi_2 \leq 2\pi$, we shall denote the set of points (ν_1, ν_2) of the real plane for which there exists a constant $A = A(\varphi_1, \varphi_2, \nu_1, \nu_2)$ such that for all nonnegative r_1, r_2 the inequality

$$|f(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2})| \leq A \exp [\nu_1 r_1^{\rho_1(r_1)} + \nu_2 r_2^{\rho_2(r_2)}]. \quad (3)$$

is satisfied.

The boundary of the set $\bar{T}(\varphi)$ is a generalization of the concept of the indicator of an entire function of one variable to the case of two variables. From Definition 3 it follows that for any $\varphi_i = \text{const}$ ($i = 1, 2$) $\bar{T}(\varphi)$ is convex and quadrant-like. For a function of one variable the basic property of the indicator is trigonometric convexity. A certain analogous assertion also holds in the case of two variables.

Denote by $K(\varphi) = K(\varphi_1, \varphi_2)$ the set of points of the real plane (ν_1, ν_2) satisfying the condition $\nu_1 \geq k_1(\varphi_1)$, $\nu_2 \geq k_2(\varphi_2)$, where $k_i(\varphi_i)$ is defined by the equality

$$k_i(\varphi_i) = \frac{h_i^{(1)} \sin \rho_i (\varphi_i^{(2)} - \varphi_i) + h_i^{(2)} \sin \rho_i (\varphi_i - \varphi_i^{(1)})}{\sin \rho_i (\varphi_i^{(2)} - \varphi_i^{(1)})},$$

$h_i^{(1)}, h_i^{(2)}$ are certain constants ($i = 1, 2$).

Theorem 4 (cf. ⁽²⁾, p. 96). Let $f(z_1, z_2)$ be an entire function with s.c.r.o. $(\rho_1(r_1), \rho_2(r_2))$, and let $T(\varphi)$ and $K(\varphi)$ be defined as above. If the point with coordinates $(h_1^{(i)}, h_2^{(i)})$ belongs to $\bar{T}(\varphi_1^{(i)}, \varphi_2^{(i)})$ for fixed $(\varphi_1^{(i)}, \varphi_2^{(i)})$, $0 \leq \varphi_i^{(2)} - \varphi_i^{(1)} \leq \pi/\rho_i$, $i = 1, 2$, then for all (φ_1, φ_2) , $\varphi_i^{(1)} \leq \varphi_i \leq \varphi_i^{(2)}$, $i = 1, 2$, the inclusion $\bar{K}(\varphi) \subset \bar{T}(\varphi)$ holds.

The well-known Borel-Pólya theorem ⁽²⁾, p. 114; ⁽⁵⁾, p. 171) asserts that the conjugate diagram of an arbitrary entire function of finite degree is the mirror image in the real axis of its indicator diagram. This theorem was generalized by M. F. Subbotin ⁽⁶⁾ and V. Bernstein ^(7,8) to entire functions of normal type of order ρ , $0 < \rho < \infty$, where instead of the indicator $h(\varphi)$ the function $\max(0, h(\varphi))$ appears. A. A. Avetisyan ⁽⁹⁾ and L. F. Lokhin ⁽¹⁰⁾ independently found this same result. A generalization of the Borel-Pólya theorem to functions of many variables of finite degree was obtained by V. K. Ivanov ^(3,4) and M. Sh. Stavskii ⁽¹¹⁾, and to functions with s.c.t. (σ_1, σ_2) for s.o. (ρ_1, ρ_2) , where $\rho_i \geq 1/2$, $0 < \sigma_i < \infty$, $i = 1, 2$, by L. I. Ronkin ⁽¹²⁾ (see also ⁽³¹⁾). V. Bernstein ^(7,8) also considered a generalization of this theorem for an entire function of

one variable with a refined order $\rho(r)$. We shall give here a generalization of his result to the case of an entire function $f(z_1, z_2)$ having s.c.r.o. $(\rho_1(r_1), \rho_2(r_2))$.

With an entire function $f(z_1, z_2)$ represented in the form of the series (1), we associate two functions

$$F_1(z_1, z_2) = \sum_{k,l=0}^{\infty} \frac{a_{kl}\gamma_1(k)\gamma_2(l)}{z_1^{k+1}z_2^{l+1}}, \quad F_2(z_1, z_2) = \sum_{k,l=0}^{\infty} \frac{a_{kl}\beta_1(k)\beta_2(l)}{z_1^{k+1}z_2^{l+1}},$$

* That is, if $(x, y) \in D$, then also $(x', y') \in D$, $x' \geq x$, $y' \geq y$.

where

$$\beta_i(t) = \int_0^{\infty} x^t \exp[-W_i(x_i)] dx_i, \quad \gamma_i(t) = \Gamma[(t + \alpha_i)\omega_i(t + \alpha_i)];$$

$W_i(z_i)$ is an analytic function for $|\arg z_i| < \pi/\rho_i$ such that $W_i(r_i e^{i\varphi_i}) \sim e^{i\varphi_i} r_i^{\rho_i} \rho_i(r_i)$ as $r_i \rightarrow \infty$, $\omega_i(x) = \ln \chi_i(x)/\ln x$, where $\chi_i(x)$ is the function inverse to $W_i(x_i)$ for $x > C = \text{const}$ and equal to some positive constant for $x \leq C$; $\alpha_i (\geq \alpha_i^0)$ is an arbitrary constant; α_i^0 is a constant depending only on the proximate order $(\rho_1(r_1), \rho_2(r_2))$ (cf. (8)).

Denote by $C_i(\varphi) = C_i(\varphi_1, \varphi_2)$, $i = 1, 2$, the set of points (ν_1, ν_2) ($\nu_1, \nu_2 > 0$) of the real plane for which the function $F_i(z_1, z_2) = F_i(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$ is analytic inside the set $\{r_j^{\rho_j} \cos \rho_j(\theta_j - \varphi_j) > \nu_j, |\theta_j - \varphi_j| \leq \min(\pi/2\rho_j, \pi), j = 1, 2\}$.

By $T^+(\varphi)$ we denote the intersection of the set $T(\varphi)$ with the first quadrant of the plane.

Theorem 5. For every entire function $f(z_1, z_2)$ having proximate order $(\rho_1(r_1), \rho_2(r_2))$, the inclusion

$$\overline{C}_1(-\varphi) \subset \overline{T}^+(\varphi) \subset \overline{C}_2(-\varphi) \quad (4)$$

holds.

If $\rho_i(r_i) \equiv \rho_i$, $i = 1, 2$, then for $\alpha_i = \rho_i$

$$F(z_1, z_2) = F_1(z_1, z_2) = F_2(z_1, z_2) = \sum_{k,l=0}^{\infty} \frac{a_{kl}\Gamma(1+k/\rho_1)\Gamma(1+l/\rho_2)}{z_1^{k+1}z_2^{l+1}},$$

and consequently $C_1(\varphi) = C_2(\varphi) = C(\varphi)$. In this case the following holds.

Theorem 6. For every entire function $f(z_1, z_2)$ having type (σ_1, σ_2) with order (ρ_1, ρ_2) ($0 < \rho_i < \infty$, $0 < \sigma_i < \infty$, $i = 1, 2$), the sets $\overline{C}(-\varphi)$ and $\overline{T}^+(\varphi)$ coincide.

We note that for $\rho_1 = \rho_2 = 1$, in the definition of the set $C(\varphi)$ the requirement of positivity of ν_1, ν_2 can be removed and the equality $\overline{C}(-\varphi) = T(\varphi)$ obtained (see (4)).

The question of the possibility of associating with an entire function a single function so that, instead of the inclusion (4), equality would hold, remains open in the general case. Under certain assumptions concerning the proximate order $(\rho_1(r_1), \rho_2(r_2))$, analogous to conditions considered by V. Bernstein, the question is answered affirmatively.

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Note: Figure translations are in progress. See original paper for figures.

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