A CRITERION FOR THE ABSOLUTE STABILITY OF MULTICONNECTED PULSE SYSTEMS WITH NONSTATIONARY CHARACTERISTICS OF NONLINEAR ELEMENTS

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Abstract

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A CRITERION FOR THE ABSOLUTE STABILITY OF MULTICONNECTED PULSE SYSTEMS WITH NONSTATIONARY CHARACTERISTICS OF NONLINEAR ELEMENTS

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A multiconnected pulse system containing a complex linear continuous part (LCP), M synchronously operating pulse elements (PE), and M nonlinear elements (NE), whose characteristics, generally speaking, depend on time, can conventionally be represented in the form of a vector block diagram (Fig. 1).

This diagram is described by the difference vector equation

$$\mathbf{x}[n] = \mathbf{f}[n] - \sum_{m=0}^{n} \mathbf{w}[n-m]\Phi(\mathbf{x}[m], m), \tag{1}$$

where $\mathbf{x}[n]$, $\mathbf{f}[n]$, $\Phi(\mathbf{x}[n], n)$ are M-dimensional vectors of the error, external action, and characteristics of the nonlinear elements, respectively; $\mathbf{w}[n]$ is a square matrix of order M, whose elements are the pulse characteristics of the linear pulse part (LPP), including the pulse elements and the complex continuous part.

Fig. 1

Assume that the LPP is stable; then the elements of the matrix $\mathbf{w}[n]$ satisfy the conditions

$$\lim_{n\to\infty} w_{sr}[n] = 0 \qquad (s, r = 1, 2, \dots, M). \tag{2}$$

Assume also that the characteristics of the nonlinear elements, for all $n \geq 0$, belong to the sectors $(a, k_{rr} - a)$, i.e., the elements of the vector $\Phi(\mathbf{x}[n], n)$ satisfy the conditions

$$\Phi(0, n) = 0; \qquad a x_r^2[n] \leq \Phi_r(x_r[n], n) x_r[n] \leq (k_{rr} - a) x_r^2[n], \tag{3}$$

where a is an arbitrarily small positive number, and $k_{rr} > 0$ are the elements of the diagonal matrix $\mathbf{K}$.

We shall say that the vector $\mathbf{f}[n]$ is vanishing if its norm

$$\|\mathbf{f}[n]\| = \max_r |f_r[n]|$$

tends to zero as n increases in such a way that

$$\sum_{n=0}^{\infty} \|\mathbf{f}[n]\| < \infty.$$

If, for any vanishing vector of external action $\mathbf{f}[n]$ and any characteristics of the nonlinear elements, in the general case nonstationary, satisfying conditions (3), the error vector is vanishing, then we shall call the multiconnected pulse system (1) absolutely stable.

The criterion of absolute stability can be formulated in the form of the following theorem:

Theorem. *In order that the multivariable pulse system (1) with a stable LTI part and with characteristics of the nonlinear elements satisfying conditions (3) be absolutely stable, it is sufficient that there exist a number* $\varkappa$ *such that*

$$\frac{1}{2}\{\mathbf{K}^{\varkappa}\mathbf{W}^*(j\bar{\omega}) + \mathbf{W}^{*T}(-j\bar{\omega})\mathbf{K}^{\varkappa}\} + \mathbf{K}^{\varkappa-1} > 0, \quad 0 \leqslant \bar{\omega} < \pi, \tag{4}$$

i.e., that the Hermitian matrix standing on the left-hand side of inequality (4) be positive definite for every $\bar{\omega} \in [0, \pi)$.

In (4), $\mathbf{W}^*(j\bar{\omega})$ is the matrix of frequency characteristics of the LTI part, defined by the relation known from the theory of the D-transform [1]

$$\mathbf{W}^*(j\bar{\omega}) = D\{\mathbf{w}[n]\}_{q=j\bar{\omega}} = \sum_{n=0}^{\infty} e^{-j\bar{\omega}n}\mathbf{w}[n], \tag{5}$$

and T denotes the transposition operation.

For the case $M = 1$, $\mathbf{W}^{*T}(j\bar{\omega}) = \mathbf{W}^*(j\bar{\omega})$, and from (4), after reduction by $K^{\varkappa}$, we obtain the known result [2]

$${}^1/_2\{W^*(j\bar{\omega}) + W^*(-j\bar{\omega})\} + K^{-1} = \operatorname{Re} W^*(j\bar{\omega}) + 1/k > 0. \tag{6}$$

Putting $\varkappa = 0$ in (4), we shall have

$$^1/_2\{\mathbf{W}^*(j\bar{\omega}) + \mathbf{W}^{*T}(-j\bar{\omega})\} + \mathbf{K}^{-1} > 0. \tag{7}$$

This result was obtained by Jury and Lee [3] on the basis of Parseval' s formula; it is a generalization of (6) to the case of multivariable systems. A similar result (although for equal sectors, $k_{rr} = k;\ r = 1, 2, \dots, \dots, M$) was first established by Halanay [4].

Let us briefly outline the proof of the theorem. In view of equation (1), form the expression

$$\sum_{n=0}^{N} \Phi^T(\mathbf{x}[n], n)\{\mathbf{K}^{\varkappa}\mathbf{x}[n] - \mathbf{K}^{\varkappa-1}\Phi(\mathbf{x}[n], n)\} =$$

$$= \sum_{n=0}^{N} \Phi^T(\mathbf{x}[n], n)\mathbf{K}^{\varkappa}\mathbf{f}[n] - \Gamma_{\varkappa}(N); \qquad (8)$$

here

$$\Gamma_{\varkappa}(N) = \sum_{n=0}^{N}\sum_{m=0}^{n} \Phi^T(\mathbf{x}[n], n)\{\mathbf{K}^{\varkappa}\mathbf{w}[n-m] + \mathbf{K}^{\varkappa-1}\sigma[n-m]\}\Phi(\mathbf{x}[m], m). \quad (9)$$

In (8) and (9), N is an integer,

$$\sigma[n-m] = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

If for all $N > 0$ the condition

$$\Gamma_{\varkappa}(N) \geqslant 0, \qquad (10)$$

is satisfied, then from (8) there follows the inequality

$$\sum_{n=0}^{N} \Phi^T(\mathbf{x}[n], n)\{\mathbf{K}^{\varkappa}\mathbf{x}[n] - \mathbf{K}^{\varkappa-1}\Phi(\mathbf{x}[n], n)\} \leqslant \sum_{n=0}^{N} \Phi^T(\mathbf{x}[n], n)\mathbf{K}^{\varkappa}\mathbf{f}[n]. \qquad (11)$$

Estimate, by means of the norms, the left-hand side of inequality (11) from below and the right-hand side from above:

$$\sum_{n=0}^{N} \Phi^T(x[n], n)\{K^{\varkappa}x[n] - K^{\varkappa-1}\Phi(x[n], n)\} \geq A\sum_{n=0}^{N} \|x[n]\|^2, \qquad (12)$$

where $A = (\max_i k_{ii})^{\varkappa-1}a^2$, and

$$\sum_{n=0}^{N}\Phi^T(x[n],n)K^{\varkappa}f[n]\leq B\sum_{n=0}^{N}\|x(n)\|, \tag{13}$$

where $B=M(\max_i k_{ii})^{\varkappa}\sup_n\|f[n]\|$. Using (12) and (13), one may replace (11) by the stronger inequality

$$A\sum_{n=0}^{N}\|x[n]\|^2\leq B\sum_{n=0}^{N}\|x[n]\|. \tag{14}$$

It follows from (14) that the norm $\|x[n]\|$ is bounded, and hence so is $\|\Phi(x[n],n)\|$. Consequently,

$$\sum_{n=0}^{N}\Phi^T(x[n],n)K^{\varkappa}f[n]<M\sum_{n=0}^{N}\|\Phi(x[n],n)\|(\max_i k_{ii})^{\varkappa}\|f[n]\|\leq$$

$$\leq M(\max_i k_{ii})^{\varkappa}\sup_n\|\Phi(x[n],n)\|\cdot\sum_{n=0}^{N}\|f[n]\|. \tag{15}$$

But since the vector $f[n]$ is vanishing, the series $\sum_{n=0}^{\infty}\|f[n]\|$ converges, and therefore the right-hand side of inequality (15), and hence also (11), is always bounded. From this we conclude that the partial sums of the series

$$\sum_{n=0}^{\infty}\Phi^T(x[n],n)\{K^{\varkappa}x[n]-K^{\varkappa-1}\Phi(x[n],n)\} \tag{16}$$

are bounded.

But, by virtue of (3), each term of this series is nonnegative; therefore the series (16) converges, and its general term tends to zero. Taking inequality (12) into account, we conclude that the error vector $x[n]$ is vanishing, and hence

$$\|x[n]\|\to 0,\qquad n\to\infty, \tag{17}$$

which proves absolute stability.

To establish conditions for the satisfiability of inequality (10), we transform it to the form

$$\Gamma_{\varkappa}(N)=\sum_{n,m=0}^{N}\Phi^T(x[n],n)\{K^{\varkappa}\hat{w}[n-m]+K^{\varkappa-1}\delta[n-m]\}\Phi(x[m],m)>0, \tag{18}$$

where

$$\hat{w}[n-m] = \frac{1}{2}\{K^{\varkappa} w[n-m] + w[m-n]K^{\varkappa}\}. \tag{19}$$

From (5), using the inversion formula (2), we have

$$w[n] = \frac{1}{2\pi}\int_{-j\pi}^{j\pi} W^*(j\omega)e^{j\omega n}\, d\omega,$$

and therefore, taking (19) into account,

$$\hat{w}[n-m] = \frac{1}{2\pi}\int_{-j\pi}^{j\pi} \frac{1}{2}\{K^{\varkappa}W^*(j\omega) + W^{*T}(-j\omega)K^{\varkappa}\}e^{j\omega(n-m)}\, d\omega. \tag{20}$$

Substituting (20) into (18), after obvious transformations we obtain

$$\Gamma_x(N) = \frac{1}{2\pi}\int_{-j\pi}^{j\pi} \sum_{n,m=0}^{N} \Psi^T(n)\left[{}^1/_2\{K^{\varkappa}W^*(j\bar{\omega}) + W^T(-j\bar{\omega})K^{\varkappa}\} + K^{\varkappa-1}\right]\Psi(m)\, d\bar{\omega}, \tag{21}$$

where

$$\Psi(n) = \Phi(x[n], n)e^{jn\bar{\omega}}. \tag{22}$$

This condition will be satisfied if the Hermitian matrix inside the square brackets is positive definite (see, for example, ([5])), which leads to condition (4).

The criterion established is readily generalized to cases involving additional constraints imposed on the characteristics of the nonlinear elements ([7]), stability of processes ([8]), etc. We shall not discuss these generalizations here. Let us note only one feature of the above criterion for the absolute stability of multiconnected systems. To each value of $\varkappa$ there correspond its own stability conditions. Unfortunately, the problem of determining the $\varkappa$ for which the broadest stability conditions for the given system are obtained is associated with considerable computational difficulties.

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CITED LITERATURE

[1] Ya. Z. Tsypkin, *Theory of Linear Pulse Systems*, Moscow, 1963.
[2] Ya. Z. Tsypkin, DAN, 145, No. 1 (1962).
[3] E. Jury, B. Lee, *Automation and Remote Control*, 26, No. 5 (1965).
[4] A. Halanay, C. R., 258, 786 (1964).
[5] Yu. A. Rozanov, *Stationary Random Processes*, Moscow, 1963.
[6] E. Jury, B. Lee, IEEE Trans. on Automatic Control, AC-9, No. 1 (1964).
[7] Ya. Z. Tsypkin, DAN, 155, No. 5 (1964).
[8] Ya. Z. Tsypkin, DAN, 152, No. 2 (1963).

Note: Figure translations are in progress. See original paper for figures.