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Abstract

Full Text

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SOME QUESTIONS IN APPROXIMATION THEORY

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MATHEMATICS

I. In the present note we shall deal mainly with four problems of approximation theory. Let X be a real Banach space with unit ball U, and let L_n be its finite-dimensional subspace.

Problem 1. Approximation of individual functions by a fixed finite-dimensional subspace. The problem consists in finding the quantity

$$E_{L_n}(f_0) = \inf_{\varphi \in L_n} \|f_0 - \varphi\|$$

of the least deviation of an element $f_0 \in X$ from L_n, and a polynomial $\varphi_0 \in L_n$ of best approximation, for which

$$E_{L_n}(f_0) = \|f_0 - \varphi_0\|.$$

Problem 2. Approximation of convex sets by finite-dimensional subspaces. Let F be a convex set lying in X. The quantity

$$\delta(F, L_n) = \sup_{f \in F} E_{L_n}(f)$$

will be called the deviation of the set F from L_n. An element $f_0 \in F$ will be called extremal if

$$\delta(F, L_n) = E_{L_n}(f_0).$$

The problem consists in determining the deviation $\delta(F, L_n)$ and the extremal elements.

Problem 3. Widths of sets. It is required to find the quantity

$$d_n(F, X) = \inf_{L_n} \delta(F, L_n) = \inf_{L_n} \inf_{\varepsilon} (F \subset L_n + \varepsilon U),$$

called the n-width of F.

Problem 4. Find

$$d^n(F) = \inf_{L^n} \inf_{\varepsilon} \{F \cap L^n \subset \varepsilon U \cap L^n\},$$

where L^n is a subspace of codimension n, i.e. a subspace of X consisting of elements f for which

$$f_1^*(f) = \dots = f_n^*(f) = 0,$$

and f_i^* are linearly independent elements of X^*. The quantity $d_n(F, X)$ was introduced by A. N. Kolmogorov. The usefulness of considering the quantity $d^n(F)$ was indicated by I. M. Gelfand.

The principal concrete results of the note concern the space C_Q of continuous real functions on a topological bicompact space Q, with metric

$$\|f\| = \max_{x \in Q} |f(x)|.$$

Sometimes the space C_Q^B on Q, with values in a Banach space B, and with norm

$$\|f\| = \max_{Q} \|f(x)\|_B$$

will be considered.

Problem 1 has been investigated in considerable detail from the general point of view in the works of Singer (see, for example, (^{1})), so that the following theorem is probably only of methodological interest, and we give it for the sake of unity of exposition.

Theorem 1. In order that an element $\varphi_0 \in L_n$ be a polynomial of best approximation to a function $f(x) \in C_Q^B$, it is necessary and sufficient that

there are $n+1$ points $x_1, \dots, x_{n+1}$ and $n+1$ functionals $f_1^*, \dots, f_{n+1}^* \in B^*$ such that

$$\sum_{k=1}^{n+1} \|f_k^*\| = 1, \qquad \sum_{k=1}^{n+1} f_k^*(f_k(x)) = E_{L_n}(f_0),$$

$$\sum_{k=1}^{n+1} f_k^*(\varphi(x_k)) = 0 \qquad \text{for all } \varphi(x) \in L_n. \tag{1}$$

From Theorem 1 there automatically follow all the criteria known to the author for polynomials of best approximation: those of Chebyshev (^{2}), Kolmogorov (^{3}), Remez (^{4}), Zukhovitskii and Krein (^{5}), Zukhovitskii and Stechkin (^{6}).

The proof of Theorem 1 is extremely simple—it is a combination of the finite-dimensional Hahn–Banach theorem and the following generalization of Shnirelman' s lemma (^{7}).

In Q there exists a system of $n+1$ points $x_1, \dots, x_{n+1}$ at which the least deviation of the function $f_0(x)$ from L_n coincides with the least deviation of $f_0(x)$ from L_n on all of Q.

Theorem 1, in a somewhat different form, is contained in (^{1}).

We pass to problem 2. It belongs to the class of minimax problems for which, apparently, there is not yet a sufficiently developed analytic apparatus. Some initial approaches to problem 2 are contained in a joint work of A. A. Milyutin and the author. Here we shall give one inconclusive necessary condition for an extremal element in the space C_Q^B.

Theorem 2. For the element $f_0(x)$ to be extremal in the set F, it is necessary that there exist $n+1$ points $x_1, \dots, x_{n+1} \in Q$ and functionals f_k^*, satisfying the conditions (1), for which, in addition, the inequality

$$\sum_{k=1}^{n+1} f_k^*(f_0(x_k) - f(x)) \geq 0 \qquad \text{for all } f \in F \tag{2}$$

holds.

In the case C_Q, the computation of $\delta(F, L_n)$ reduces to the finite-dimensional problem of finding

$$\max \ \max_{f \in F} \sum_{k=1}^{n+1} \mu_k f(x_k), \tag{2'}$$

where the first max is taken over all points $x_1, \dots, x_{n+1}$ and all real collections $\mu_1, \dots, \mu_{n+1}$ for which

$$\sum_{k=1}^{n+1} |\mu_k| = 1, \qquad \sum_{k=1}^{n=1} \beta_k \varphi(x_k) = 0 \qquad \text{for all } \varphi \in L_n.$$

It seems to us interesting to make a more detailed investigation of problem $(2')$, in particular, to determine what the standard methods of the calculus of variations can give here.

We pass to the problem on the widths d_n. Let us also introduce the value of the *linear* width:

$$d_n'(F, X) = \inf_{L_n} \inf_{\Lambda \colon F \to L_n} \|f - \Lambda f\|,$$

where Λ is a linear operator mapping F into L_n.

Theorem 3. Among all Banach spaces, only Hilbert spaces possess the property that always

$$d_n(F; X) = d_n'(F, X) \geq d^n(F), \qquad d_n(F, X) = d_n(F, X'), \qquad F \in X \in X'.$$

The first examples of the noncoincidence of d_n and d'_n and of the decrease of d_n when F is immersed in an enveloping space were indicated in (^{9}).

II. We now pass to some concrete results. Everywhere below the space X is $C_{[-1,1]}$.

Theorem 4. For any integers $n \geq 0$ and $r > 0$ there exists, and is uniquely determined on $[-1, 1]$, a function $f_{n,r}(x)$ having the following property:

1. It assumes its maximum and minimum values alternately at $n + r + 1$ points.
2. Its r-th derivative is equal in absolute value to one, has exactly n changes of sign, and $f_{n,r}(+1) = +1$.

For any integers $n \geq 0$, $r > 0$, and real β, $|\beta| > 1$, there exists, and is uniquely determined on the whole line, a function $f_{n,r,\beta}(x)$ satisfying the conditions:

1. It assumes its maximum and minimum values on the segment $[-1, 1]$ alternately at $n + r$ points.
2. Its r-th derivative is equal in absolute value to one and has there exactly n changes of sign.
3. $f^{(r)}_{n,r,\beta}(x) \equiv 1$ for $x \geq 1$; $f^{(r)}_{n,r,\beta}(x) \equiv f^{(r)}_{n,r,\beta}(-1)$ for $x \leq -1$; $f'_{n,r,\beta}(\beta) = 0$.

The functions $f_{n,r}(x)$ generalize the Chebyshev polynomials

$$f_{0,r}(x) = \frac{1}{r!} T_r(x) = \frac{1}{r!\, 2^{r-1}} \cos(r \arccos x).$$

The functions $f_{n,r,\beta}(x)$ generalize Zolotarev polynomials, coinciding with them also when $n = 0$. The functions $f_{n,r}(x)$ and $f_{n,r,\beta}(x)$ are solutions of a number of extremal problems connected with differential operators on the segment $[-1, 1]$. We indicate some of them.

The problem of widths of differentiable functions on a segment. Denote by W_r the class of functions $f(x)$, defined on $[-1, 1]$, for which

$$\sup_{x \in [-1,1]} \operatorname{vrai} |f^{(r)}(x)| \leq 1. \tag{3}$$

Theorem 5*.

$$d_n(W_r) = d^n(W_r) = \begin{cases} \infty, & n \leq r - 1, \\ \|f_{n-r,r}\|. \end{cases}$$

The space best approximating the set W_r is the space $L_{n,r}$ of dimension n, consisting of functions $f(x) = p(x) + \Phi(x)$, where $p(x)$ is a polynomial of degree

$\leq r-2$, and the $r-1$ derivatives of the function $\Phi(x)$ are piecewise constant on the intervals $\Delta_k = [\xi_k, \xi_{k+1}]$, where ξ_k are the break points of $f^{(r)}_{n-r,r}(x)$.

In order to approximate, with the necessary accuracy, a function $f(x) \in W_r$ by functions from L_{nr}, it is enough to interpolate it by functions from L_{nr} at the n points $\eta_1, \dots, \eta_n$, which are the zeros of $f_{n-r,r}(x)$. The space $L_{r+1,r}$ is not the only extremal one; for $n > r+1$ the question has not been clarified.

In connection with Theorem 5 it seems of interest to us to point out the following fact.

Consider the class $\widetilde{W}_2$ of functions $f(x) \in [-1,1]$ satisfying inequality (3) and periodic together with all derivatives up to order $r-1$. The widths $d_{2n-1}(\widetilde{W}_r)$ were computed in (8). The extremal space turned out to be the subspace T_{2n-1} of trigonometric polynomials of degree $\leq n-1$. It turns out that for no n is it is the unique extremal subspace. Namely, the following holds.

Theorem 6. Apart from the space T_{2n-1}, the extremal space for the class $\widetilde{W}_r$ among $(2n)$-dimensional spaces is the space $L_{2n,r}$ of functions $f(x)$ whose $(r-1)$-st derivative is piecewise constant on the intervals $\Delta_k = [\xi_k, \xi_{k+1}]$, where ξ_k are the zeros of the function $\sin nx$. In this case

$$d_{2n-1}(\widetilde{W}_r) = d_{2n}(\widetilde{W}_r).$$

For $r = 1$ this result was noted in (8).

The problem on inequalities for derivatives, i.e., the problem of computing

$$\max_{W_r \cap \alpha U} \|f^{(k)}(x)\| = F(k,\alpha); \quad \alpha > 0; \quad 0 < k < r$$

(U is the unit sphere in $C[-1,1]$), or, in other words, the problem of finding

$$F(k,\alpha) = \max_{\substack{\|f\| \leq \alpha \\ f \in W_r}} \|f^{(k)}\|.$$

On the whole line this problem was solved by A. N. Kolmogorov (9). On a finite interval the following holds.

Theorem 7. For every α there exist unique $n_0(\alpha,k)$ and $\beta_0(\alpha,k)$ such that

$$F(k,\alpha) = \|f^{(k)}_{n_0,r,\beta_0}\|.$$

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* In the case $r = 2$ this result was also obtained by R. L. Frum-Ketkov.

Theorem 7 generalizes V. A. Markov' s inequality for polynomials.

The problem on inequalities for derivatives at a fixed point of the interval $[-1,1]$.

Theorem 8. For any point $\xi \in [-1,1]$ there exist $n_0(\xi,k)$, $\beta_0(\xi,k)$ such that

$$\max_{\substack{f\in W_r \\ \|f\|\le\alpha}} |f^{(k)}(\xi)| = |f^{(k)}_{n_0,r,\beta_0}(\xi)|.$$

The problem of finding $\max_{|\xi|>1} |f(\xi)| = G(\xi,\alpha)$ **over all** $f(x)$, for which $\left|f^{(r)}(x)\right| \le 1$, $\|f\| = \max_{x\in[-1,1]} |f(x)| \le \alpha$.

Theorem 9. There exist $n_0(\xi,\alpha)$, $\beta_0(\xi,\alpha)$ such that $G(\xi,\alpha) = f_{n_0,r,\beta_0}(\xi)$.

Theorem 9 also generalizes the analogous property of Chebyshev polynomials.

I consider it my duty to note the great influence that conversations with A. A. Milyutin had on me, and to express to him my gratitude for this.

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Note: Figure translations are in progress. See original paper for figures.