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Abstract

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR PARABOLIC EQUATION

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In the present note the results of the paper (^{5}) are applied to the study of the asymptotic behavior of solutions of a parabolic equation. Consider, in a bounded domain Ω with $2m$-times smooth boundary S of real n-dimensional space R_n $(n \geq 2)$, the elliptic operator

$$\mathcal{L}(x, D) = \sum_{|\beta| \leq 2m} a_\beta(x) D^\beta, \qquad \beta = (\beta_1, \dots, \beta_n),$$

$$|\beta| = \beta_1 + \dots + \beta_n, \quad D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}, \quad D_i = \partial / \partial x_i.$$

We assume the leading coefficients to be continuous in $\overline{\Omega}$, the remaining ones measurable and bounded. Let differential operators $B_j(x, D)$, $j = 1, \dots, m$, of orders $m_j < 2m$, with coefficients from $C_{2m-m_j}(S)$, be given on S. We shall assume B_j to be subject to the complementing condition (^{2}), and $\mathcal{L}$, for $n = 2$, to satisfy the root condition (^{2}). In addition, the system B_j will be regarded as normal, i.e. the m_j are distinct and S is not characteristic for B_j at any point. Consider the following problem:

$$\frac{\partial u(t, x)}{\partial t} = \mathcal{L}(x, D) u + F(t, x, u, Du, \dots, D^k u), \qquad t \geq t_0, \tag{1}$$

$$B_j u = 0, \quad j = 1, \dots, m; \qquad u(t_0, x) = u_0(x), \quad 0 \leq k \leq 2m - 1.$$

Here $F(t, x, u, \dots, D^k u)$ is defined and continuous in the totality of variables for $t \geq t_0$, $x \in \overline{\Omega}$, $-\infty < u, \dots, D^k u < \infty$, and satisfies, with respect to the variables $u, \dots, D^k u$, a Lipschitz condition with constant $\gamma(t)$; $\gamma(t)$ is continuous and nonnegative for $t \geq 0$. Recently obtained results (^{1}) make it possible to apply to (1) the methods of semigroup theory. The operator $\mathcal{L}$ and the boundary operators B_j generate in $L_p(\Omega)$ $(p > 1)$ a closed linear operator A_p with domain of definition

$$D(A_p) = W_p^{2m}(\Omega; B_j),$$

the set of functions from $W_p^{2m}(\Omega)$ satisfying the boundary conditions in the sense of S. L. Sobolev.

Let the following conditions be fulfilled:

1. $$(-1)^m \frac{\mathcal{L}'(x,\xi)}{|\mathcal{L}'(x,\xi)|} \neq e^{i\theta} \quad \text{for } x \in \overline{\Omega},\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
   and for any real
   $$\xi = (\xi_1, \dots, \xi_n) \neq 0.$$

2. If $x \in S$, ν is the normal to S at the point x, $\xi \neq 0$ and is parallel to the tangent at the point x to S, and $t_k^+(\xi,\lambda)$ are the m roots of the polynomial
   $$(-1)^m \mathcal{L}'(x, \xi + t\nu) - \lambda$$
   with positive imaginary part, where λ is any number on the ray $\arg\lambda = \theta$, then the polynomials
   $$B_j'(x, \xi + t\nu), \qquad j = 1, \dots, m,$$
   are linearly independent modulo the polynomial
   $$\prod_{k=1}^{m} (t - t_k^+(\xi,\lambda)).$$
   Here
   $$\mathcal{L}'(x,D),\ B_j'(x,D)$$
   are the principal parts of $\mathcal{L}(x,D)$, $B_j(x,D)$.

From the results of (1) it follows that, when conditions 1, 2 are satisfied, A_p generates in $\mathcal{L}_p(\Omega)$ an analytic semigroup $e^{A_p t}$, $t > 0$, strongly continuous at $t = 0$. Moreover $R(\lambda; A_p)$ is completely continuous, and the spectrum of A_p is discrete and does not depend on p. Let λ_0 be a real number, $\lambda_0 > \sup \operatorname{Re} \sigma(A_p)$, where $\sigma(A_p)$ is the spectrum of A_p. For the operator $\lambda_0 I - A_p$, fractional powers $(\lambda_0 I - A_p)^\alpha$, $\alpha \geqslant 0$, are defined.

Theorem 1. *Let $p > 1$, k be an integer, $0 \leqslant k \leqslant 2m-1$, $(2m-k)p > n$, $\alpha = \dfrac{k+n/p}{2m}$. In that case, for any $\varepsilon > 0$ the operator*

$$(\lambda_0 I - A_p)^{-\alpha-\varepsilon}$$

acts completely continuously from $\mathcal{L}_p(\Omega)$ into $C_k(\overline{\Omega})$.

The method of proof of the theorem is analogous to the method of (3) and uses one embedding theorem of V. P. Il' in (4) and an a priori estimate in $\mathcal{L}_p(\Omega)$ for solutions of the elliptic equation (2). The function $F(t, x, u, \dots, D^k u)$ generates in $\mathcal{L}_p(\Omega)$ a nonlinear operator $F(t,u) = F(t, x, u, \dots, D^k u)$; moreover, under the conditions of Theorem 1, $F(t,u)$ is defined for $t \geqslant 0$, $u \in D(A_p^{\alpha+\varepsilon})$, $\varepsilon > 0$, and $\Phi(t,u) = F(t, A_p^{-\alpha-\varepsilon})$ is defined for all $t \geqslant 0$, $u \in \mathcal{L}_p(\Omega)$, is continuous (in the metric of $\mathcal{L}_p(\Omega)$) with respect to (t,u), and satisfies the condition

$$\|\Phi(t,u_1) - \Phi(t,u_2)\|_{\mathcal{L}_p(\Omega)} \leqslant k(\varepsilon)\gamma(t)\|u_1 - u_2\|_{\mathcal{L}_p(\Omega)}.$$

We shall call a solution of the integral equation in the Banach space $\mathcal{L}_p(\Omega)$

$$u(t) = e^{A_p(t-t_0)}u_0 + \int_{t_0}^{t} e^{A_p(t-\tau)}F(\tau, u(\tau))\,d\tau. \tag{2}$$

a generalized solution of problem (1).

Here $u_0 \in \mathcal{L}_p(\Omega)$, and $u(t)$ are functions for $t \geqslant t_0$ with values in $\mathcal{L}_p(\Omega)$, i.e. $u(t) = u(t,x)$, where $u(t,x)$, for each fixed t, is an element of $\mathcal{L}_p(\Omega)$. The existence and uniqueness of the generalized solution for any $u_0 \in \mathcal{L}_p(\Omega)$ are proved in ([5]). The generalized solution is k times continuously differentiable with respect to x for $t > t_0$, $x \in \Omega$.

Let λ_s, $s = 1, 2, \dots$, be the eigenvalues of the operator A_p; $\mu_s = \operatorname{Re}\lambda_s$; let d_s be all the distinct values among the μ_s, numbered so that $d_s > d_{s+1}$. The only possible accumulation point of the d_s is $-\infty$. Denote by $\mathcal{L}(\{d_j\}_{j=1}^r)$ the subspace of $\mathcal{L}_p(\Omega)$ consisting of the eigenvectors and associated vectors of the operator A_p corresponding to eigenvalues λ with $\operatorname{Re}\lambda = d_j$ for some $j = 1, \dots, r$. Then $\mathcal{L}_p(\Omega)$ can be represented as the direct sum of two subspaces reducing A_p:

$$\mathcal{L}_p(\Omega) = M(\{d_j\}_{j=1}^r) + \mathcal{L}(\{d_j\}_{j=1}^r).$$

Definition 1. A transformation f acting from $M(\{d_j\}_{j=1}^r)$ into $\mathcal{L}_p(\Omega)$ will be called **regular** if it is continuous and, for any $u \in M(\{d_j\}_{j=1}^r)$, the projection fu onto $M(\{d_j\}_{j=1}^r)$ coincides with u. The image $M(\{d_j\}_{j=1}^r)$ will be called a **regular image** of $M(\{d_j\}_{j=1}^r)$.

Definition 2. Let $u(t,x)$ be defined for sufficiently large t and $x \in \Omega$. We shall call the exponent of u in $\mathcal{L}_p(\Omega)$

$$\omega_p(u) = \lim_{t\to\infty} \frac{\ln \|u(t,x)\|_{\mathcal{L}_p(\Omega)}}{t}.$$

Analogously, the exponent of u in $C_k(\overline{\Omega})$ is

$$\omega_{C_k}(u) = \lim_{t\to\infty} \frac{\ln \|u(t,x)\|_{C_k(\overline{\Omega})}}{t}.$$

We assume that the right-hand sides of the equalities make sense.

Theorem 2. *Let an integer $r \geqslant 1$, $\varepsilon > 0$,*

$$\varepsilon < \min_{1\leqslant i\leqslant r} \frac{d_i - d_{i+1}}{2},$$

and numbers p, k, m, n be given satisfying the conditions of Theorem 1,

$$\alpha \in \left(\frac{k+n/p}{2m}, 1\right).$$

There exists a $\delta = \delta(\varepsilon, r, \alpha) > 0$ such that, if $\gamma(t)$ satisfies the condition

$$\int_{t_0}^{t} \frac{e^{(\varepsilon/2)(\tau-t)}\gamma(\tau)}{(t-\tau)^\alpha}\, d\tau, \quad \int_{t}^{\infty} e^{(\varepsilon/2)(t-\tau)}\gamma(\tau)\, d\tau < \delta, \quad t > t_0,$$

then for any generalized solution of problem (1) either $\omega_p(u) < d_{r+1} + \varepsilon$, or $\omega_p(u)$ lies in one of the ε-neighborhoods of the points d_i, $i = 1, \dots, r$. Moreover, $\omega_p(u) = \omega_{C_k}(u)$, and the set of initial functions $u_0(x)$ for which the exponents of the corresponding solutions do not exceed $d_i + \varepsilon$, $1 \le i \le r+1$, is a regular image $M(\{d_j\}_{j=1}^{i-1})$.

Corollary. Suppose there exist $\varepsilon_n \to 0$, $T_n \to \infty$, $\beta_n \to 0$ such that

$$\int_{T_n}^{t} \frac{e^{(\varepsilon_n/2)(\tau-t)}\gamma(\tau)}{(t-\tau)^\alpha}\, d\tau, \quad \int_{t}^{\infty} e^{(\varepsilon_n/2)(t-\tau)}\gamma(\tau)\, d\tau < \beta_n, \quad t > T_n, \quad n = 1, 2, \dots.$$

In this case the exponent in $\mathcal{L}_p(\Omega)$ (and hence u in $C_k(\overline{\Omega})$) of any generalized solution of problem (1) coincides with one of d_i, $i = 1, 2, \dots$.

The conditions of the corollary are satisfied if $\lim_{t\to\infty} \gamma(t) = 0$ or $\gamma(t) \in \mathcal{L}_q(t_0, +\infty)$, $q > \dfrac{1}{1-\alpha}$.

We shall consider equation (1) in $\mathcal{L}_2(\Omega)$ and assume additionally that the operator $\mathcal{L}$ and the boundary operators are formally self-adjoint. In this case, instead of requiring conditions 1, 2 to hold, it is sufficient to require that they hold for $\theta = 0$, and then A_2 is the self-adjoint semibounded-from-above operator (1).

The following refinement of Theorem 2 is valid:

Theorem 3. Let an integer $r \ge 1$, $\varepsilon > 0$, $\varepsilon < \min_{1\le i\le r} \dfrac{d_i - d_{i+1}}{2}$, $2(2m-k) > n$, $\alpha \in \left(\dfrac{k+n/2}{2m}, 1\right)$, be given, and suppose that

$$\varphi_i(t) = \begin{cases} \alpha^\alpha \displaystyle\int_{t_0}^{t} \frac{e^{\varepsilon(\tau-t)}\gamma(\tau)\, d\tau}{(t-\tau)^\alpha}, & t \in \left[t_0, t_0 + \dfrac{\alpha}{\lambda_0 - d_i}\right], \\ \alpha^\alpha \displaystyle\int_{t-\frac{\alpha}{\lambda_0-d_i}}^{t} \frac{e^{\varepsilon(\tau-t)}\gamma(\tau)\, d\tau}{(t-\tau)^\alpha} + (\lambda_0 - d_i)^\alpha \int_{t_0}^{t-\frac{\alpha}{\lambda_0-d_i}} e^{\varepsilon(\tau-t)}\gamma(\tau)\, d\tau, & t > t_0 + \dfrac{\alpha}{\lambda_0 - d_i}, \end{cases}$$

$$\psi_i(t) = (\lambda_0 - d_{i-1})^\alpha \int_t^\infty e^{\varepsilon(t-\tau)}\gamma(\tau)\,d\tau,$$

$$R_i = \sup_{t>t_0} \varphi_i(t), \qquad Q_i = \sup_{t>t_0} \psi_i(t), \qquad i = 1, \dots, r+1, \quad Q_0 = 0,$$

then

$$R_i + Q_i < 1, \quad i = 1, \dots, r+1.$$

In this case the assertions of Theorem 2 are valid for $p = 2$.

The results obtained can be applied to the study of the stability of the zero solution of (1).

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Note: Figure translations are in progress. See original paper for figures.