UNIQUENESS, EXISTENCE, AND A PRIORI PROPERTIES OF GENERALIZED SOLUTIONS

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Abstract

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UNIQUENESS, EXISTENCE, AND A PRIORI PROPERTIES OF GENERALIZED SOLUTIONS

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An extensive literature is devoted to the problem of existence and uniqueness of generalized solutions of boundary-value problems and to the clarification of the functional properties of these solutions; however, comparatively complete results have been obtained only for the case of the first boundary-value problem (see, for example, ([1,2]), where several different definitions of generalized solutions of boundary-value problems in the linear case are also given).

In the present note we formulate some new theorems on the existence and uniqueness of weak generalized (in the sense indicated below) solutions of the third boundary-value problem for a general linear parabolic equation of second order:

$$u_t' = L(u) + f(P,t), \tag{1}$$

and also indicate a priori functional properties of these solutions (in one case important for applications).

We introduce the following notation: Ω is a bounded domain of the n-dimensional Euclidean space of points $P = \{x_1, x_2, \dots, x_n\} \in R^n$; Σ is its boundary; $Q(T) = \Omega \times (0,T)$ is a cylinder in the product space $R^n \times (-\infty < t < +\infty) \equiv R^{n+1}$; $S(T) \equiv \Sigma \times (0,T) \subset R^{n+1}$ is its lateral surface.

1. We shall assume that the operator $L(u) \equiv L_1(u) + L_2(u)$ has divergent principal part

$$L_1(u) = \sum_i \sum_j \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u}{\partial x_j}\right),$$

where

$$L_2(u) = \sum_j b_j \frac{\partial u}{\partial x_j} + cu,$$

$$\sum_i \sum_j A_{ij}\alpha_i\alpha_j \geq \mu \sum_j \alpha_j^2,$$

where $\mu > 0$ is a constant independent of the point $(P,t) \in R^{n+1}$. We further assume that the coefficients A_{ij}, b_j, c are bounded measurable functions in the cylinder $Q(T)$, that $a(P,t)$ is a nonnegative bounded measurable function on $S(T)$, and

$$f(P,t) \in L_2(Q(T)), \qquad \varphi(P) \in L_2(\Omega), \qquad v(P,t) \in L_2(S(T)). \tag{2}$$

2. **Definition.** By a weak generalized solution of the third boundary-value problem for the parabolic equation (1) we shall call

call a pair of functions $\{\psi(P,t), u(P,t)\}$, $\psi(P,t) \in L_2(S(T))$, $u(P,t) \in L_2(Q(T))$, $u'_{x_j}(P,t) \in L_2(Q(T))$ (u'_{x_j} is the Sobolev generalized derivative of the function u), satisfying almost everywhere in $(0,T)$ the integral identity

$$\int_\Omega u\Phi\Big|_0^t d\Omega - \int_{Q(t)} u\Phi'_t\,dQ + \int_{Q(t)} \sum_{ij} A_{ij}u'_{x_j}\Phi'_{x_i}\,dQ - \int_{Q(t)} \sum_i b_i u'_{x_i}\Phi\,dQ -$$

[
_{Q(t)} cu ,dQ

{Q(t)} f ,dQ + {S(t)} (au-v) ,dS =0,]

whatever the function $\Phi \in W_2^1(Q(T))$. Here, by definition, it is assumed that $u|_{t=0} = \varphi(P)$, $u|_{S(T)} = \psi(P,t)$. This specification of the extension of the concept of solution is caused mainly by conditions (2), which are inevitable in many applied problems.

Existence theorem. Suppose that the assumptions of item 1 are satisfied. Suppose, moreover, that Σ, the boundary of Ω, is a piecewise smooth surface or consists of a finite number of such (nonintersecting) surfaces. In the general case Σ may be regarded as belonging to the class A^{2*}.

Then there exists at least one weak generalized solution of the third boundary-value problem for equation (1), satisfying the energy inequality

$$\int_\Omega u^2\Big|_0^t\,d\Omega + \int_{Q(t)} \sum_j (u'_{x_j})^2\,dQ + \int_{S(t)} u^2\,dS \leq c(T)\left\{\int_{Q(t)} f^2\,dQ + \int_{S(t)} v^2\,dS\right\}$$

almost everywhere in $(0,T)$; here also

$$u|_{t=0} = \varphi(P), \qquad u|_S = \psi(P,t).$$

The constant $c(T)$ increases monotonically to $+\infty$ as $T \to \infty$.

The question of uniqueness of this solution is clarified by the following theorem.

Uniqueness theorem. Let (ψ_1, u_1), (ψ_2, u_2) be two arbitrary solutions of the third boundary-value problem; if u_1, u_2 are continuous as elements of the space $L_2(\Omega)$ on the closed interval $[0,T]$, then they coincide: $u_1 \equiv u_2$ almost everywhere on $Q(T)$.

Under the same continuity condition, the "boundary" values ψ are uniquely determined by the function u, namely: ψ is the weak limit in $L_2(S(T))$ as $\delta \to 0$ of the values u on the pieces $S^*_\delta = \Sigma^*_\delta \times [0,T]$, where δ is the modulus of the shift vector of the piece $\Sigma^* \subset \Sigma$ into the domain Ω.

If u is continuous in $t \in [0,T]$ as an element of the space $W^1_2(\Omega)$, then ψ is the strong limit in $L_2(S)$ of the values $u|_{S^*_\delta}$ as $\delta \to 0$.

Thus, if one includes in the definition of generalized solution the requirement of continuity in t of the function $u(P,t)$ as an element of the space $L_2(\Omega)$, then one obtains an unconditional uniqueness theorem.

Differential properties of generalized solutions. Here we shall consider only one case important for applications, when the functions f, φ, v are bounded.

Theorem 1. Let $f(P,t)$, $\varphi(P)$, $v(P,t)$ be bounded measurable functions on their domains of definition. Suppose, moreover, that the conditions of item 1 are satisfied. Then, if the weak generalized solution of the third boundary-value problem is unique, then u is equivalent to a function continuous in $Q(T)$

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* A^2 is the class of surfaces representable (in local coordinates) by continuous functions from W^1_2.

(denote it again by u), for which the a priori estimates

$$\|u\|_{Q(T)} \equiv \sup_{(P,t)\in Q} |u| \leqslant C(M),$$

$$\|u\|^{\alpha}_{\overline{Q}^*(T)} = \sup_{(P_1,t_1),(P_2,t_2)\in\overline{Q}^*} \frac{|u(P_1,t_1) - u(P_2,t_2)|}{\|P_1 - P_2\|^{\alpha} + (t_1 - t_2)^{\alpha/2}} \leqslant C(M, \overline{Q}^*), \qquad (3)$$

and also

$$\|\psi\| \equiv \operatorname*{vrai\,max}_{(P,t)\in S(T)} |\psi| \leqslant C(M),$$

hold, where $0 < \alpha \leqslant 1$; $\overline{Q}^* \in R^{n+1}$ is an arbitrary cylinder belonging to $\overline{Q} - \Gamma$; $\Gamma = S(T) \cup \Omega \times (t = 0)$; the constant $C(M)$ depends only on the maximum M of the moduli $|b_i|$, $|c|$, $|f|$, $|\varphi|$, $|v|$, $|a|$; $C(M, \overline{Q}^*)$ depends on the same maximum M and on the distance of the cylinder $\overline{Q}^*$ from Γ.

It follows from the estimates (3) that u, as an element of the space $L_2(\Omega)$, is a continuous function of time t.

Theorem 2. Suppose that the hypotheses of Theorem 1 are satisfied, and also the conditions:

a) $\operatorname{vrai\,max} |A'_{ijx_k}| \leqslant C$;

b) $A_{ij} \in C^0(Q(T))$, $k, i, j = 1, \dots, n$.

Then for the solution (ψ, u), in addition to (3), the estimates

$$\|u'_{x_j}\|_{\overline{Q}^*} = \sup_{(P,t) \in \overline{Q}^*} |u'_{x_j}| \leqslant C(M, \overline{Q}^*);$$

$$\|u'_{x_j}\|^{\alpha}_{\overline{Q}^*} \leqslant C(M, \overline{Q}^*). \tag{4}$$

will also hold.

Under conditions a), b), the a priori estimates (4) guarantee the continuity of the conormal derivatives

$$\frac{\partial u}{\partial \gamma} \equiv \sum_i \sum_j A_{ij} \frac{\partial u}{\partial x_i} \cos(\mathbf{n}, \mathbf{x}_j),$$

computed inside $\overline{Q} - \Gamma$, and also the validity of the third boundary condition in the following sense: let $\Omega_m \to \Omega$, $m \to \infty$, $\overline{\Omega}_m \subset \Omega$ be an arbitrary sequence of domains with smooth boundaries Σ_m; then, for any $\Phi \in W_2^1(Q)$, the limiting equality

$$\lim_{m \to \infty} \int_{S_m} \left(\frac{\partial u}{\partial \gamma}\right)_m \Phi \, ds = -\int_S (a\psi - v) \Phi \, dS,$$

holds, where

$$\left(\frac{\partial u}{\partial \gamma}\right)_m = \sum_i \sum_j A_{ij} \frac{\partial u}{\partial x_i} \cos(\mathbf{n}_m, \mathbf{x}_j),$$

$\mathbf{n}_m$ is the exterior normal to Σ_m.

In proving the theorems of the present article, certain results of the works (^{1,2}) were used.

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Note: Figure translations are in progress. See original paper for figures.