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Abstract

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SOME RESULTS ON INFINITESIMAL BENDINGS OF SURFACES "LOCALLY"AND "GLOBALLY"

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1. By a "bending" we shall always mean an infinitesimal bending.

2. In the article ([1]) we established theorems describing the local structure of Darboux surfaces. In the present paper we give some simple applications of the results of ([1]) to questions of bending surfaces "locally" and "globally." For notation and terminology, see ([1]).

3. Let $R \in C^2$ and have positive curvature. The following uniqueness theorems for bendings hold (in the formulations of the theorems, by U we mean any infinite set having a limit point M inside R; by C we mean an arbitrary constant vector).

Theorem 1. Let $\mathbf{y} = C$ on the set U. Then the corresponding bending is trivial.

Theorem 2. Let $s = z - [\mathbf{yr}] = C$ on the set U. Then the corresponding bending is trivial.

Theorem 3. Suppose that at the point $M \in R$ the condition $(\mathbf{yn}) \neq 0$ is satisfied and let $\mathbf{y} \parallel C \neq 0$ on the set U. Then the corresponding bending is trivial.

Theorem 4. Let the surface R be analytic and let $\mathbf{y} \parallel C \neq 0$ on the set U. Then the corresponding bending is trivial, with the possible exception of the case when U coincides with an arc of the line $L : (C\mathbf{n}) = 0$, passing through the point M and corresponding to the line of shadow on R for the direction C.

We note that the possible exception indicated in Theorem 4 is realized in examples of nontrivial bendings.

4. In this paragraph and below it is assumed that $R \in C^{2,\alpha}$, $\alpha > 0$, and has positive curvature.

Definition. We say that a bending $\mathbf{z}$ of a surface R with boundary is a sliding bending relative to the plane Π with guiding normal $\mathbf{l}$ (or a sliding bending perpendicular to the direction $\mathbf{l}$), if on the boundary of R the condition $(\mathbf{zl}) = 0$ is satisfied.

Proceeding from the local structure of Z, the following theorems are proved.

Theorem 5. Let the line of shadow for the direction $\mathbf{l}$ pass through a point M lying on the convex surface R. Then, no matter how small a neighborhood $\bar{R} \subset R$ of the point M is taken, inside $\bar{R}$ there is a domain $R_1 \subset \bar{R}$ such that R_1 admits a sliding bending perpendicular to the direction $\mathbf{l}$.

Theorem 6. There exists an uncountable set of surfaces of negative curvature or of surfaces with an isolated flattening point such that in an arbitrarily small neighborhood X of any of their points there is a domain $\bar{X} \subset X$ that is projected one-to-one onto some plane Π and admits a sliding bending relative to Π.

Theorem 6 is a supplement to a theorem of N. V. Efimov ([2]) stating that almost all analytic surfaces with a flattening point of sufficiently high order admit no analytic bendings even "locally."

Theorem 6 asserts that among the remaining surfaces, of a set of measure zero, there are infinitely many surfaces that admit a bending "in the small" even with an additional boundary condition.

5. Since the surfaces Y and S, according to (1), have no supporting planes at interior points, $|\mathbf{y}|$ and $|\mathbf{s}|$ attain their maximum only on the boundary of the surface.* It follows that a closed ovaloid is rigid, since, in view of the absence of a boundary, there are no points at which the functions $|\mathbf{y}|$ and $|\mathbf{s}|$ would have a maximum.
6. On the structure of Darboux surfaces "in the large" one can prove the following theorems.

Theorem 7. *Let the surface R be projected one-to-one onto some plane Π, and suppose that the normal to $\bar{R} = R + \Gamma$, Γ being the boundary of R, is nowhere parallel to Π. Then, by adding a trivial rotation, the bending diagram Z can be made an everywhere regular surface.*

Conversely, if R contains a closed line of shadow for some direction (and therefore R is not projected one-to-one onto any plane), then the bending diagram Z always has an edge, i.e. Z is an irregular surface.

Theorem 7 explains why the existence of a bending of the surface R under certain boundary conditions depends on the one-to-one projectability of R onto some plane.

Theorem 8. *Let the surface R be strictly convex "in the large," and let it be visible from the origin on one side. Then the surface H is also strictly convex "in the large," and is also visible on one side from the origin.*

For the remaining Darboux surfaces, the following general theorem holds.

Theorem 9. *Let the surface R be projected one-to-one onto some plane, and let $\mathbf{z}$ be the field of its bending. If the surface R is visible from the origin on one side, then the field $\mathbf{z}$ can be changed, by adding a suitably chosen trivial bending, so that all Darboux surfaces for the altered field become everywhere regular (except for the branch points of the surfaces Y, S, Q, and Ω, which are preserved).*

7. We shall state several results on bendings with certain boundary conditions.

Theorem 10. *Let the surface R be projected one-to-one onto some plane Π. Then R (even if it is multiply connected) admits no bending for which the condition*

$$(\mathbf{yn}) = 0$$

is satisfied on the boundary R.

Remark 1. If R contains a closed line of shadow for some direction, then one can indicate examples of simply connected convex surfaces admitting a bending with the condition $(\mathbf{yn}) = 0$ on the boundary. Such surfaces, for example, can be constructed in the following way. Take a spherical segment larger than a hemisphere and admitting Rembs bendings, under which the boundary of the segment slides in its own plane. Then, by adding to $\mathbf{y}$ a sufficiently large trivial rotation $\mathbf{y}_0$, perpendicular to the plane of the boundary of the segment, one can arrange that the line $(\mathbf{yn}) = 0$ does not intersect the boundary and consists of one closed simple component L_0. That part of the chosen segment which has the line L_0 as its boundary will be the desired surface.

Remark 2. With the aid of surfaces admitting a bending with the condition $(\mathbf{yn}) = 0$ on the boundary, one can construct a convex surface R with the following property: R admits a nontrivial bending with the condition of stationarity of the distance from the boundary to some point M. Here the surface R is visible from M on one side, but between the point M and the surfac–

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* We note that, independently and somewhat earlier than the author, functions of this kind were found by M. I. Voidekhovskii, on the basis of other considerations, as he reported at the seminar on geometry "in the large" at Moscow University.

it is impossible to draw a plane separating them by R. This example shows that the assertion of Theorem 5 of ([4]) may be false when either of the two conditions of this theorem is violated (that this theorem may be false under the condition that the surface R is visible from different sides from the point M follows from the example given in ([5])).

Theorem 11. Let $\Gamma \subset C^2$ be the edge of the surface R (possibly multiply connected), and let U be the set of all points of Γ at which the tangent plane to R contains the direction $\mathbf{I}$ (i.e. $(\mathbf{In}) = 0$ at the points of U). If, at least at one point $M \in U$, the tangent to Γ is not parallel to $\mathbf{I}$, then R admits no bendings by sliding perpendicularly to the direction $\mathbf{I}$, under the additional condition that $(\mathbf{yn}) \neq 0$ at all points of U*.

Theorem 12. Let the edge of the surface R be a plane curve, possibly consisting of several components lying in different planes. Then R admits no bendings by sliding if, for each component of the edge, the sliding occurs perpendicularly to the direction of some vector lying in the plane of this component.

It turns out that, under the conditions of Theorem 12, the bendings of R under consideration satisfy the boundary condition $d\mathbf{y} \parallel d\mathbf{r}$, and hence, by virtue of the well-known theorem of N. V. Efimov from ([6]), the validity of Theorem 12 follows.

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[6] N. V. Efimov, UMN, **3**, issue 2, 47 (1948).

* If $(\mathbf{yn}) \neq 0$ at some of the points of U, then it is sufficient to require nonparallelism of $\mathbf{I}$ to the tangent to the edge R only at these points of U.

Note: Figure translations are in progress. See original paper for figures.