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Abstract

Full Text

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ON THE APPROXIMATION OF PERIODIC FUNCTIONS IN THE MEAN

(Presented by Academician A. N. Kolmogorov, January 7, 1965)

Let $f(x)$ be a 2π-periodic function with Fourier–Lebesgue series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx);$$

and

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

be the partial sum of order n. J. Favard ([1]) estimated the deviation of a function from its partial sum in the metric of the space of continuous periodic functions in terms of the corresponding deviation of the r-th derivative of the function:

$$\|f - S_{n-1}(f)\|_C \le K_r n^{-r} \|f^{(r)} - S_{n-1}(f^{(r)})\|_C, \qquad n = 1, 2, \ldots; \quad r = 1, 2, \ldots \quad (1)$$

Moreover, if

$$\varphi_r(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin\{(2k-1)x - \pi \cdot 2^{-1} r\}}{(2k-1)^{r+1}},$$

then inequality (1) becomes an equality for the functions $f(x) = n^{-r}\varphi_r(nx)$.

Let

$$T_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos kx + \beta_k \sin kx)$$

be an arbitrary trigonometric polynomial of order n with real coefficients α_k and β_k, and

$$E_n(f, C) = \inf_{T_n} \|f - T_n\|_C.$$

As Sun Yung-shen ([2]) observed, by the same arguments one can also prove another inequality:

$$E_{n-1}(f, C) \leq K_r n^{-r} E_{n-1}(f^{(r)}, C), \tag{2}$$

where equality is also attained for the function $\varphi_{r,n}(x)$.

There are many generalizations of inequalities (1) and (2) in the metric of the space C. We shall be interested in analogues of these inequalities in the various metrics of the spaces L_p with norm

$$\|f\|_{L_p} = \left\{ \frac{1}{\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}, \qquad p \geq 1.$$

In the metric of the space L_1, inequality (1) was obtained by N. Akhiezer ([3]).

Theorem 1. For any $p \geqslant 1$ and any natural $r \geqslant 1$ and $n \geqslant 1$, the inequality

$$\|f - S_{n-1}(f)\|_{L_p} \leqslant \|\varphi_r\|_{L_p} n^{-r} \|f^{(r)} - S_{n-1}(f^{(r)})\|_C,$$

holds, and the equality sign is attained for $f(x) = \varphi_{r,n}(x)$.

In proving Theorem 1, J. Favard' s inequality (1) and a result of A. N. Kolmogorov (4) (Theorem II) are used.

Let $p \geqslant 1,\ p' \geqslant 1,\ q = p(p-1)^{-1},\ q' = p'(p'-1)^{-1}$,

$$K_\perp(n, p, p', r) = \sup_{\|f^{(r)} - S_{n-1}(f^{(r)})\|_{L_{p'}} \leqslant 1} \|f - S_{n-1}(f)\|_{L_p},$$

$$K_e(n, p, p', r) = \sup_{E_{n-1}(f^{(r)}, L_{p'}) \leqslant 1} E_{n-1}(f, \alpha_p).$$

In this notation the following is valid.

Theorem 2. For any natural $r \geqslant 1,\ n \geqslant 1$, we have

$$K_\perp(n, p, p', r) = K_e(n, q', q, r).$$

From Theorems 1 and 2 it follows that

Theorem 3. For any $p \geqslant 1$ and any natural $r \geqslant 1,\ n \geqslant 1$, the inequality

$$E_{n-1}(f, L_1) \leqslant \|\varphi_r\|_{L_q} n^{-r} E_{n-1}(f^{(r)}, L_p) \qquad (q = p(p-1)^{-1},\ p \geqslant 1),$$

holds, and the equality sign is attained for $p > 1$ for a certain function $f(x) = f_*(x, n, p, r)$.

Finally, let us note two special cases:

$$K_e(n, 1, \infty, r) = K_\perp(n, 1, \infty, r) = \frac{4}{\pi} K_{r+1} n^{-r},$$

$$K_e(n, 1, 2, r) = K_\perp(n, 2, \infty, r) = 2 \left\{ \frac{K_{2r+1}}{\pi} \right\}^{1/2} n^{-r}.$$

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Note: Figure translations are in progress. See original paper for figures.