# ON THE THEORY OF OPERATORS OF THE FORM $\dfrac{d}{dt}-A$

MATHEMATICS

1965

SovietRxiv

## Abstract

## Full Text

UDC 517.944

**MATHEMATICS**

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# ON THE THEORY OF OPERATORS OF THE FORM $\dfrac{d}{dt} - A$

*(Presented by Academician I. M. Vinogradov on 3 III 1965)*

We shall consider the equation

$$Lu \equiv \frac{du}{dt} - Au = f \tag{L}$$

with boundary condition

$$\mu_1 u\big|_{t=0} - \mu_2 u\big|_{t=a} = 0, \tag{$\Gamma$}$$

where, for $t \in [0, a]$, the functions $u(t), f(t)$ take values in a complex Hilbert space $\mathfrak{H}$; $\mu_1, \mu_2 : \mathfrak{H} \to \mathfrak{H}$ are bounded linear operators, and $A : \mathfrak{H} \to \mathfrak{H}$ is an unbounded normal operator commuting with $d/dt$. Equation (L) is a very popular object of investigation, but in this case it is usually assumed either that $\mu_2 = 0$ and $A$ is the infinitesimal generator of a semigroup, or else one of the indicated assumptions is removed (for example, in ([1],[2])). In ([3]) it was shown that the use of conditions of the form ($\Gamma$) makes it possible to find "good" problems also for "pathological" operators $L$. In the present article the same question is considered from a more general point of view. We shall be interested in two types of conditions ($\Gamma$). In the first (problems L[Γ-I]) $\mu_2 = 1$ and $\mu_1 = \mu$ is a complex number. Conditions of this type make it possible to describe resolvable extensions ([3]) of the operator $L$ in the case when the spectrum of the operator $A$ does not fill the whole complex plane $C$, and include the "classical" problems. In conditions of the second type (problems L[Γ-II]) $\mu_1 = \mu^-(A)$, $\mu_2 = \mu^+(A)$ are special projection operators constructed from the operator $A$. These conditions make it possible to describe a resolvable extension of the operator $L$ also for an operator $A$ with spectrum filling the whole plane $C$. It should be noted that the results described pertain to a rather special class of normal operators $A$: to differential operators with constant coefficients on the $n$-dimensional torus.

Let us pass to the consideration of problems of the first type. If $A$ is a numerical parameter, it is not difficult to write down an explicit formula giving the solution of problem L[Γ-I]. Introduce the notation:

$$S_t \equiv S(A,t) = e^{tA}; \qquad \int_\alpha^\beta e^{(t-\tau)A} f(\tau)\, d\tau = I_\alpha^\beta(t,A) f.$$

Then the desired formula (under the assumption that $\mu - S_a \neq 0$) is written in the form:

$$u(t) = L^{-1} f(t) = (\mu I_0^t(t,A) + I_t^a(t,A) S_a)(\mu - S_a)^{-1} f. \tag{1}$$

If now $A$ is a normal operator (i.e. $A = R + iQ$, where $R, Q$ are commuting self-adjoint operators), formula (1) still retains its meaning under the assumption that $\mu$ is not a point of the discrete spectrum of the operator $S_a$.

Let $T$ be an $n$-dimensional torus obtained by identifying opposite faces of a cube with edges of length $2\pi$, referred to coordinates $(x_1, \dots, x_n)$. On $T$ are defined a linear manifold $P$ of sufficiently smooth functions and the complex Hilbert space $H_x$ of functions with summable-

square with the usual scalar product and norm. If $A(s)$ is a polynomial with constant complex coefficients

$$A(s) = \sum_{|\alpha| \le m} A_\alpha s^\alpha, \qquad s^\alpha = s_1^{\alpha_1} \dots s_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \dots + \alpha_n,$$

then $\dot{A} \equiv \dot{A}(-iD)$ is the corresponding differential operator, defined on $P$, and $\dot{A}(-iD) e^{is\cdot x} = A(s) e^{is\cdot x}$. By the operator $A : H_x \to H_x$ we mean the closure of $\dot{A}$ in $H_x$.

The subsequent exposition contains a number of more or less obvious propositions, the proofs of which are omitted.

**Proposition 1.** The operator $A$ is normal, i.e. $AA^* = A^*A$.

Denote by $\mathfrak{S}$ the set of integer vectors $(s_1, \dots, s_n)$, where each $s_i$ independently ranges over the set of values $0, \pm 1, \pm 2, \dots$, and by $A(\mathfrak{S})$ the set of values of the polynomial $A(s)$ for $s \in \mathfrak{S}$.

**Proposition 2.** The spectrum $\sigma A$ of the operator $A$ consists of the closure in $C$ of the set $A(\mathfrak{S})$.

In considerations on $T$, the role of the operational calculus that gives meaning to formula (1) is played by the use of the classical Fourier series.

Let now $H = H_x \otimes H_t$, where $H_t$ is the Hilbert space $\mathcal{L}_2$ of functions defined on the interval $[0,a]$. By the operator $L : H \to H$ we shall mean the closure in $H$ of the operator of the left-hand side of equation (L), defined on smooth functions satisfying the conditions (Γ). Or: a function $u \in H$ belongs to $\mathfrak{D}_L$ if there exist sequences $\{u_i\}$ of smooth functions satisfying the conditions (Γ), and a function $f \in H$, such that $|u - u_i, H| \to 0$, $|Lu_i - f, H| \to 0$ as $i \to \infty$. The corresponding solution of problem L-Γ will be called a strong solution, or simply a solution. The operator $L$ is **regular** if $L^{-1}$ exists, is bounded, and is defined on all of $H$. In what follows, unless otherwise stated, the notation $\mu = \infty$ is used. It is assumed that in this case the problem in question is that for which $\mu_1 = 1,\ \mu_2 = 0$.

The set of values $\mu$ for which the operator $L$ is not regular will be called the **$\mu$-spectrum** of the problem LΓ and denoted by $\Sigma(A, a)$. By $\sigma A,\ \sigma S(A, a),\ \sigma L(A, a, \mu)$ we denote the spectra of the operators $A,\ S_a,\ L$. The investigation of problem LΓ is the study of various relations and dependencies among the spectra listed.

**Theorem 1.** *The equality*

$$\Sigma(A, a) = \sigma S(A, a)$$

*holds.*

The proof is quite elementary, but somewhat long. We give the main steps.

**Proposition 3.** The spectrum $\sigma S(A, a)$ is the closure in $C$ of the set

$$\exp[aA(\mathfrak{S})],$$

which forms the discrete spectrum $\sigma_d S(A, a)$ of the operator $S_a$.

Denote by $\Sigma_d(A, a)$ the set of points $\mu$ of the plane $C$ such that for $\mu \in \Sigma_d(A, a)$ the operator $L^{-1}$ does not exist.

**Proposition 4.** The equality

$$\Sigma_d(A, a) = \sigma_d S(A, a)$$

is valid.

**Lemma 1.** *If $\mu \notin \sigma S(A, a)$, then the operator $L$ is regular.*

The proof is based on the validity, in this case, of the energy inequality

$$|u, H| \leq c\,|Lu, H|, \tag{$\Phi$}$$

established by the methods used in ([3]).

**Lemma 2.** *If*

$$\mu \in \sigma_c S(A, a) \equiv \sigma S(A, a) \setminus \sigma_d S(A, a),$$

*then the operator $L^{-1}$ exists, is defined on a set dense in $H$, and is unbounded.*

The existence of the operator $L^{-1}$ is a trivial consequence of the uniqueness of the Fourier coefficients; unboundedness is established by considering a sequence $\{f^k\}_1^\infty$ of right-hand sides

$$f^k = \exp(is^k \cdot x),$$

where the sequence $\{s^k\}_1^\infty$ of vectors from $\mathfrak{S}$ is such that

$$|\mu - \exp[aA(s^k)]| \to 0$$

as $k \to \infty$. From Propositions 3 and 4 and Lemmas 1, 2, Theorem 1 follows.

In considering $\sigma L(A, a, \mu)$ it is convenient to introduce the notation $A_\lambda = \lambda E + A$.

**Proposition 5.** For given $A, a, \mu$, a point $\lambda \in \sigma_d L(A, a, \mu)$ (or $\lambda \in \sigma_c L(A, a, \mu)$) if and only if $\mu \in \sigma_d S(A_\lambda, a)$ (or $\mu \in \sigma_c S(A_\lambda, a)$).

The following propositions clarify the special position of the Cauchy problem among the LG-I problems.

**Proposition 6.** If the point 0 (the point $\infty$) belongs to $\sigma S(A, a)$, then $\sigma L(A, a, 0)$ (respectively $\sigma L(A, a, \infty)$) fills the whole plane $C$. If $\mu \neq 0, \infty$, then $\sigma L(A, a, \mu)$ fills the whole plane if and only if the spectrum $\sigma S(A, a)$ fills the whole plane.

**Proposition 7.** The points $0, \infty$ cannot belong to $\Sigma_d(A, a)$.

Thus, in the compact case under consideration (the manifold $T \times [0, a]$ is compact), the uniqueness theory for the Cauchy problem is trivial. It follows from Proposition 6 that, if $\sigma S(A, a)$ does not fill $C$, one can always choose $\mu$ so that the LG-I problem will determine a solvable extension of the operator $L$.

**Lemma 3.** *For $n < 3$ the spectrum $\sigma A$ cannot fill the whole plane $C$; for $n \geqslant 3$ there exist operators $A$ whose spectrum fills the whole plane.*

The verification of the first part of the assertion is elementary. The second part is established by considering the operator for which

$$A(s) = s_1 + \alpha s_2 + i(s_3 + \beta s_2^2),$$

where $\alpha, \beta$ are irrational. Indeed, the fractional parts of the pairs

$$\{\alpha y, \ \beta y^2\}, \qquad y = 0, \pm 1, \pm 2, ...$$

are uniformly distributed in the unit square ([4]).

Thus, for $n \geqslant 3$ the LG-I problem does not allow one to describe a solvable extension of the operator $L$ for arbitrary $A$.

Let us pass to the LG-II problem. Let the operator $A$ be fixed and

$$A(s) = r(s) + iq(s),$$

where $r, q$ are real polynomials. We divide the set $\mathfrak{S}$ into two subsets $\mathfrak{S}^-$, $\mathfrak{S}^+$, putting $s \in \mathfrak{S}^-$ if $r(s) \leqslant 0$ and $s \in \mathfrak{S}^+$ if $r(s) > 0$. The partition of $\mathfrak{S}$ induces a decomposition of $H_x$ into the sum of orthogonal subspaces

$$H_x = H_x^- \oplus H_x^+,$$

where $H_x^-$ ($H_x^+$) is the closed linear span of the set of vectors of the form $\exp(is \cdot x)$, $s \in \mathfrak{S}^-$ ($s \in \mathfrak{S}^+$). Let $\mu^-, \mu^+$ be the projection operators in $H_x^-, H_x^+$, respectively.

**Theorem 2.** *For any operator $A$ with constant coefficients on $T$, the LG-II problem in which $\mu_1 = \mu^-$, $\mu_2 = \mu^+$, determines a solvable extension of the operator $L$.*

For the proof it is enough to note that the coefficients of the expansion

$$u(x,t) = \sum_{\mathfrak{S}} u_s(t) e^{is \cdot x}$$

of an arbitrary smooth solution of the described problem will be determined from the equations

$$u_s(t) = \int_0^t e^{(t-\tau)A(s)} f_s(\tau)\, d\tau, \qquad s \in \mathfrak{S}^-,$$

$$u_s(t) = -\int_t^a e^{(t-\tau)A(s)} f_s(\tau)\, d\tau, \qquad s \in \mathfrak{S}^+,$$

where $f_s(\tau)$ are the coefficients of the corresponding expansion of the right-hand side. An estimate of the form

$$|u_s(t)|^2 \leqslant |f_s(t) H_t|^2 \int_0^t e^{2(t-\tau)r(s)}\, d\tau$$

in the case $s \in \mathfrak{S}^-$, and an analogous estimate for the case $s \in \mathfrak{S}^+$, make it possible (by virtue of the corresponding sign of $r(s)$) to establish the energy inequality-

property ($\Phi$), whence the existence and uniqueness theorem for the solution for every $f \in H$ follows from (3) in the usual way.

The choice of the operators $\mu_1, \mu_2$ in problem LG-II, when describing the solvable extension, can of course be varied within fairly broad limits, but the use in the considerations of subspaces of the type $\mathfrak{S}^-, \mathfrak{S}^+$ (for $\sigma A = C$) seems to be dictated by the essence of the matter.

We note in conclusion that the approach described to the study of equation (L) induces a certain special classification of differential operators $A$ with constant coefficients and of boundary-value problems LG-I according to the spectral properties of the operator $S_a$. The methods used evidently admit generalizations also to equations more general than (L).

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Received
18 II 1965

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*Note: Figure translations are in progress. See original paper for figures.*