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Abstract

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ON THE SUMMABILITY OF FOURIER SERIES IN EIGENFUNCTIONS OF THE LAPLACE OPERATOR BY CESÀRO, RIESZ, AND POISSON-ABEL MEANS

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In the present paper we study Fourier series in eigenfunctions of the equation $\Delta u + \lambda u = 0$ in an arbitrary N-dimensional domain g and with arbitrary boundary conditions ensuring the existence of a countable set of nonnegative eigenvalues $\{\lambda_n\}$ and a complete orthonormal system of eigenfunctions $\{u_n(x)\}$*.

The central result of the present paper is the following

Main theorem. *If a function $f(x)$ belongs to the class $L_2(g)$, then the Fourier series of this function is summable almost everywhere in the domain g to this function: 1) by Cesàro means of any positive order, 2) by the Poisson–Abel method, 3) by normal Riesz means $R(\lambda_n, \alpha)$ of order α, where α is any number satisfying at least one of the inequalities $\alpha \geqslant 1,\ \alpha > (N-1)/4$.**

On the question under study, the mathematical literature contains only the results of B. M. Levitan ([1]) and E. C. Titchmarsh (([2]), Ch. 18), asserting that if a function $f(x)$ belongs to the class $L_2(g)$ and, moreover, is continuous at the given point x at least in some generalized sense, then the Fourier series of $f(x)$ is summable at this point by normal Riesz means $R(\lambda_n, \alpha)$ of order $\alpha > (N-1)/2$.

We outline the scheme of the proof of the main theorem. Along the way a number of results of independent interest will be indicated.

1°. The principal place in the paper is occupied by the following assertion.

Theorem 1. *If a function $f(x)$ belongs to the class $L_2(g)$, then the Fourier series of this function is summable almost everywhere in the domain g to this function by normal Riesz means $R(\lambda_n, \alpha)$ of order $\alpha > (N-1)/4$.*

The proof of Theorem 1 is preceded by the proof of several lemmas.

* Particular cases of the Fourier series studied are multiple trigonometric Fourier series, Fourier series in eigenfunctions of the first, second, or third homogeneous boundary value problems in an arbitrary bounded N-dimensional domain g. We emphasize that we have sufficiently only generalized functions from $W_2^{(1)}(g)$ among the eigenfunctions, so that the boundary of the N-dimensional domain g is not subject to any smoothness requirements.

** A numerical series $\sum_{k=1}^{\infty} a_k$ is said to be summable to the number s:

1) by Cesàro means of order α, if

$$\lim_{n\to\infty}\left\{\frac{n!}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\sum_{k=1}^{n}a_k\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n+1-k)}{(n+1-k)!}\right\}=s;$$

2) by the Poisson-Abel method, if

$$\lim_{r\to 1-0}\sum_{k=1}^{\infty}a_k r^{k-1}=s;$$

3) by normal Riesz means $R(\lambda_n,\alpha)$ of order α, if

$$\lim_{\lambda\to\infty}\sum_{\lambda_k<\lambda}a_k\left(1-\frac{\lambda_k}{\lambda}\right)^{\alpha}=s.$$

Lemma 1. Let $\{\gamma_k\}$ be some bounded sequence, $\lambda>0$, and let ρ be any number in the segment $1\leq\rho\leq\sqrt{\lambda}$. Then the estimate

$$\int_g\left|\sum_{|\sqrt{\lambda_k}-\sqrt{\lambda}|\leq\rho}u_k(x)u_k(y)\gamma_k\right|dy=O\left(\sqrt{\lambda}^{(N-1)/2}\rho\right),\qquad(1)$$

holds uniformly with respect to x in any closed subdomain g' lying strictly inside the domain g.

For the proof of Lemma 1 it suffices to apply to the integral standing on the left-hand side of (1) Bunyakovsky' s inequality and to use the orthonormality property of the eigenfunctions and the preliminary asymptotic formula established in § 1, Chapter I of [3].

Lemma 2. For any $\alpha>(N-1)/4$ and for all real numbers λ satisfying the condition $\lambda\geq\bar{\lambda}>0$, the set of functions*

$$w_\lambda(x)=\int_g\left|\sum_{\lambda_k<\lambda}u_k(x)u_k(y)\left(1-\frac{\lambda_k}{\lambda}\right)^{2\alpha}\right|dy\qquad(2)$$

is bounded uniformly with respect to x in any closed subdomain g' lying strictly inside the domain g.

For the proof of Lemma 2 we fix an arbitrary strictly interior subdomain g' of the domain g, a point x in g', and a positive number R not exceeding the minimum distance between the boundaries of the domains g and g'. Taking the function

$$v(r) = \begin{cases} \dfrac{\Gamma(2\alpha+1)\cdot 2^{2\alpha}\lambda^{(N-4\alpha)/4}}{(2\pi)^{N/2}} \dfrac{J_{N/2+2\alpha}(r\sqrt{\lambda})}{r^{N/2+2\alpha}}, & \text{for } r < R, \\ 0, & \text{for } r \geq R, \end{cases}$$

which is the principal term of the Riesz spectral function, we compute the k-th Fourier coefficient v_k of this function by direct integration, using the mean-value theorem for the eigenfunctions of the equation $\Delta u + \lambda u = 0$. Using the arguments given on page 129 of [5], we obtain for this Fourier coefficient the expression

$$v_k = \delta_k u_k(x) \left(1 - \frac{\lambda_k}{\lambda}\right)^{2\alpha} - \frac{\Gamma(2\alpha+1)\cdot 2^{2\alpha}\lambda^{(N-4\alpha)/4}}{\lambda_k^{(N-2)/4}} u_k(x) I_k, \tag{3}$$

where

$$I_k = \int_R^\infty J_{N/2+2\alpha}(r\sqrt{\lambda}) J_{N/2-1}(r\sqrt{\lambda_k}) r^{-2\alpha}\, dr, \tag{4}$$

and $\delta_k = 1$ for $\lambda_k < \lambda$ and zero for the remaining indices k. Next both parts of (3) are multiplied by $u_k(y)$, and after this summed over all indices k from 1 to $n+p$, where n is the largest of the indices k for which $\lambda_k < \lambda$, while p is any sufficiently large natural number. Passing to the moduli and carrying out integration over the domain g in the coordinates of the point y, we arrive at the inequality

$$w_\lambda(x) \leq \int_g \left| \sum_{k=1}^{n+p} v_k u_k(y) \right| dy + \Gamma(2\alpha+1)\cdot 2^{2\alpha} \int_g \left| \sum_{k=1}^{n+p} \lambda^{(N-4\alpha)/4} \frac{u_k(x) u_k(y)}{\lambda_k^{(N-2)/4}} I_k \right| dy. \tag{5}$$

* From the point of view of the theory of orthogonal series, the set $\{w_\lambda(x)\}$ is the collection of the so-called Lebesgue functions for the Riesz summation method (see, for example, Chapter 3 of [4]).

It is quite clear that the first of the integrals on the right-hand side of (5), for large n, differs arbitrarily little from the integral $\int_g |v(r)|\, dr$, whose boundedness (for $x \in g'$) is verified directly. Thus, in order to prove the boundedness of the set of functions (2), it is enough to establish that the second integral on the right-hand side of (5) is bounded by a constant independent of the numbers n and p. This is done by means of rather delicate arguments using the above-mentioned Lemma 1, the lemma proved on p. 245 of the book ([2]), and a scheme, in principle close to that set forth in § 3 of the paper ([5]).

Lemma 3. Let $F(x)$ be an arbitrary function of the class $L_2(g)$,

$$\alpha > \frac{N-1}{4}, \quad S_\lambda(x) = \sum_{\lambda_k<\lambda} F_k u_k(x) \left(1 - \frac{\lambda_k}{\lambda}\right)^\alpha, \quad \overline{S}_\Lambda(x) = \sup_{\lambda\le\Lambda} |S_\lambda(x)|.$$

Then for a strictly interior subdomain g' there exists a constant M such that

$$\int_{g'} \overline{S}_\Lambda(x)\,dx \le M \left[\sum_{\lambda_k<\Lambda} F_k^2\right]^{1/2}. \tag{6}$$

The scheme of the proof of Lemma 3 is very close to the well-known Kolmogorov-Plessner scheme (see, for example, ([6]), pp. 335–336, the passage from (2.9) to (2.10)). In the proof, the above-established Lemma 2 is used essentially.

Lemma 3 makes it possible to prove Theorem 1 formulated above. We put

$$\sigma_\lambda(x) = \sum_{\lambda_k<\lambda} f_k u_k(x) \left(1 - \frac{\lambda_k}{\lambda}\right)^\alpha,$$

$$J_\mu(x) = \lim_{\nu\to\infty} \int_{g'} \sup_{\mu\le\lambda\le\nu} [\sigma_\lambda(x) - \sigma_\mu(x)]\,dx,$$

$$j_\mu(x) = \lim_{\nu\to\infty} \int_{g'} \inf_{\mu\le\lambda\le\nu} [\sigma_\lambda(x) - \sigma_\mu(x)]\,dx$$

and prove that

$$\lim_{\mu\to\infty} J_\mu = \lim_{\mu\to\infty} j_\mu = 0.$$

For this purpose, for each $\mu > 0$ one fixes a number n such that $\lambda_n^{N+1} \le \mu$, and estimates the difference

$$\sigma_\lambda(x) - \sigma_\mu(x) = \sum_{\lambda_k<\lambda_n} f_k u_k(x) \left\{\left(1 - \frac{\lambda_k}{\lambda}\right)^\alpha - \left(1 - \frac{\lambda_k}{\mu}\right)^\alpha\right\} +$$

$$+ \sum_{\lambda_n<\lambda_k<\lambda} f_k u_k(x) \left(1 - \frac{\lambda_k}{\lambda}\right)^\alpha - \sum_{\lambda_n<\lambda_k<\mu} f_k u_k(x) \left(1 - \frac{\lambda_k}{\mu}\right)^\alpha. \tag{7}$$

The first term on the right-hand side of (7) is estimated very simply, proceeding from the fact that the expression in braces is $O(1/\lambda_n^N)$. The second and third terms on the right-hand side of (7) are estimated with the aid of Lemma 3. (Here inequality (6) is written for the function $F(x)$, for which $F_1 = F_2 = \cdots = F_n = 0$, and all the remaining Fourier coefficients coincide with the corresponding Fourier coefficients of $f(x)$.) Thus Theorem 1 is proved.

2°. To prove the main theorem, besides Theorem 1, we shall need two more assertions concerning general Fourier series with respect to an arbitrary complete orthonormal system in the domain g. Denote by $\{\nu_n\}$ any increasing sequence of indices satisfying the condition* $1 < q \leqslant \lambda_{\nu_{n+1}}/\lambda_{\nu_n} \leqslant r$, and by $f(x)$ an arbitrary function of the class $L_2(g)$.

Theorem 2. *The convergence of the subsequence of partial sums of the Fourier series $S_{\nu_n}(x)$ almost everywhere in the domain g is a necessary condition for the summability of the Fourier series almost everywhere in the domain g by normal Riesz means $R(\lambda_n, \alpha)$ of any positive order α.*

Theorem 3. *The convergence of the subsequence $S_{\nu_n}(x)$ almost everywhere in the domain g is a sufficient condition for the summability of the Fourier series almost everywhere in g by normal Riesz means $R(\lambda_n, \alpha)$ of order $\alpha \geqslant 1$.*

Theorems 2 and 3 are in fact proved on pp. 146 and 147 of the book ([4]). It remains only to note that, for any $\alpha > 0$,

$$1 - \left(1 - \frac{\lambda_k}{\lambda}\right)^{\alpha} = O\left(\frac{\lambda_k}{\lambda}\right) \quad (\text{for } \lambda_k \leqslant \lambda).$$

From Theorems 1, 2, and 3 it follows that, for any function $f(x)$ of the class $L_2(g)$, the Fourier series is summable almost everywhere in g to $f(x)$ by normal Riesz means $R(\lambda_n, \alpha)$ of order $\alpha \geqslant 1$.

Furthermore, from the asymptotics of the eigenvalues it follows that, for any function $f(x)$ of the class $L_2(g)$, the subsequence of partial sums of the Fourier series $S_{2^n}(x)$ converges almost everywhere in g. But this ensures the summability of the Fourier series almost everywhere in the domain by Cesàro means of any positive order and by the Poisson–Abel method (see ([7]), Theorems 5.8.3 and 5.8.5).

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* Here by $\{\lambda_n\}$ one may understand any nondecreasing unbounded sequence of nonnegative numbers.

Note: Figure translations are in progress. See original paper for figures.