Reports of the Academy of Sciences of the USSR

1965

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1965. Volume 162, No. 3

MATHEMATICS

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ON CLOSED IDEALS IN A CERTAIN COUNTABLY NORMED ALGEBRA OF ENTIRE ANALYTIC FUNCTIONS

(Presented by Academician I. G. Petrovskii on 9 XII 1964)

Consider the set $\mathfrak{F}$ of all entire analytic functions $f(z)$ of order of growth $\rho \leq 1$ and, in the case $\rho = 1$, of minimal type $\sigma = 0$. This means that every $f(z)$, for any $\varepsilon > 0$, satisfies the inequality

$$|f(z)| \leq Ce^{\varepsilon|z|},$$

where the constant $C = C(f, \varepsilon)$. In what follows it is assumed that the smallest of the possible values of C is taken.

Obviously, $\mathfrak{F}$ is an algebra over the field of complex numbers; addition, multiplication by a number, and multiplication of elements of $\mathfrak{F}$ among themselves are defined naturally as the corresponding operations on the functions $f(z)$. Introduce in $\mathfrak{F}$ a system of norms depending on the parameter $\varepsilon > 0$ and defined by the formula

$$\|f\|_\varepsilon = C(f, \varepsilon).$$

The norms increase as ε decreases; we obtain a space with a countably normed topology ([1]), convergence meaning convergence in each of the norms. Let us note that, without changing the topology, one may restrict the variation of ε to only a countable sequence $\varepsilon_i \to 0$. In the topology introduced by us, as is easily verified, the algebraic operations in $\mathfrak{F}$ are continuous with respect to their arguments.

An **ideal** $J \subseteq \mathfrak{F}$ is a linear subspace that is stable under multiplication by any $f \in \mathfrak{F}$:

$$Jf \subseteq J.$$

We shall call an ideal **trivial** if $J = 0$ or $J = \mathfrak{F}$. An ideal J is called **closed** if it is a closed set in $\mathfrak{F}$ (or, what in the present case is the same, if from $f_i \to f$ and $f_i \in J$ it follows that $f \in J$).

Our task is the **description of all nontrivial closed ideals** $J \subset \mathfrak{F}$.

Theorem 1. Let $b_1, b_2, \ldots$ be any sequence of complex numbers (possibly with repetitions), arranged in nondecreasing order of moduli,

$$|b_1| \leq |b_2| \leq \cdots \tag{1}$$

and satisfying the single condition

$$\frac{1}{n}|b_n| \xrightarrow[n\to\infty]{} \infty. \tag{2}$$

(In the case of a finite sequence, this condition, of course, is omitted.)

Then the set of all possible functions $f(z)$ that have among their zeros all $b_1, b_2, \ldots$ (with multiplicity not less than their multiplicity in the sequence):

1) forms a nontrivial closed ideal in $\mathfrak{F}$, and moreover
2) every nontrivial closed ideal $J \subset \mathfrak{F}$ can be obtained in this way.

Let us note that, generally speaking, a closed ideal J is not principal, i.e. there does not exist a function $f_0(z) \in \mathfrak{F}$ for which $b_1, b_2, \ldots$ would precisely exhaust its zeros and by which, consequently, all functions $f(z) \in J$ would be divisible.

For the existence of $f_0(z)$ it is necessary and sufficient to impose an additional condition on the sequence $b_1, b_2, \ldots$: the convergence (possibly not absolute) of the series $\sum_n b_n^{-1}$.

It follows from Theorem 1 that every maximal ideal among the nontrivial closed ideals corresponds to the case of a one-term sequence b_1 and consists of all possible functions $f(z) \in \mathfrak{F}$ vanishing at the point b_1; such an ideal will be maximal also among all nontrivial ideals.

The countably normed algebra $\mathfrak{F}$ considered by us is the simplest special case of the associative hyperenvelope $\mathfrak{F}(G_r)$ of the Lie algebra G_r over the field of complex numbers ([2]), namely, in our case $\mathfrak{F} = \mathfrak{F}(G_1)$ (the case when the dimension of the Lie algebra $r = 1$). The analogous problem of describing closed ideals in the general case $\mathfrak{F}(G_r)$, apparently, should be of considerable interest for the theory of infinite-dimensional representations of Lie groups; however, it is very difficult.

We can associate with the countably normed space $\mathfrak{F}$ its conjugate space $\mathfrak{F}'$, which can be identified with the space of all analytic germs $\varphi(u)$ of one complex variable u in a neighborhood of $u = 0$.

The passage to the limit in $\mathfrak{F}'$ is defined by the condition $\varphi_i(u) \to \varphi(u)$, where the sequence $\varphi_i(u)$ tends uniformly to $\varphi(u)$ in at least one neighborhood of the point $u = 0$. In $\mathfrak{F}'$ the strongest topology compatible with this passage to the limit is introduced, i.e. closed sets are defined as those closed with respect to passage to the limit along sequences.

Every function $f(z) \in \mathfrak{F}$ is interpreted in $\mathfrak{F}'$ as the continuous operator $f(\partial)$, where $\partial = d/du$.

If

$$f(z) = \sum_n \alpha_n \frac{z^n}{n!} \in \mathfrak{F}, \qquad \varphi(u) = \sum_n \beta_n u^n \in \mathfrak{F}',$$

then

$$f(\partial)\varphi(u) = \sum_n \alpha_n \varphi^{(n)}(u);$$

the series converges in the topology of $\mathfrak{F}'$.

The spaces $\mathfrak{F}$ and $\mathfrak{F}'$ are conjugate in the following sense: if one introduces the scalar product

$$(f, \varphi) = f(\partial)\varphi(u)\big|_{u=0},$$

then, for arbitrarily fixed $\varphi \in \mathfrak{F}'$, it gives the most general form of a continuous linear functional in $\mathfrak{F}$ (different from 0 when $\varphi \neq 0$), and for arbitrarily fixed $f \in \mathfrak{F}$—analogously in $\mathfrak{F}'$.

Between nontrivial closed ideals $J \subset \mathfrak{F}$ and nontrivial ($\neq 0, \mathfrak{F}'$) closed subspaces $E' \subset \mathfrak{F}'$, invariant with respect to the operator $\partial = d/du$, there naturally arises a one-to-one correspondence according to the principle

$$f(\partial)\varphi(u) = 0, \tag{*}$$

where $f \in J,\ \varphi \in E'$; here E' consists of all φ satisfying $(*)$ for

for any $f \in J$ and, analogously, J consists of all f satisfying $(*)$ for any $\varphi \in E'$.

Everything said concerning $\mathfrak{F}'$ follows as a special case from the results of [2]. If we add Theorem 1 to this, we obtain

Theorem 2. *The general form of a nontrivial closed subspace $E' \subset \mathfrak{F}'$, invariant with respect to the operator $\partial = d/du$, can be described as follows: it is the minimal closed subspace spanned by the germs*

$$\varphi(u) = e^{bu}, \quad ue^{bu}, \dots, \quad u^{k-1}e^{bu}, \tag{**}$$

where b runs through a sequence of complex values

$$b = b_1, b_2, \dots, \quad |b_1| \le |b_2| \le \dots,$$

k denotes the multiplicity of the given member of the sequence b; moreover, the only condition imposed on the sequence (in the case of its infinitude) is

$$\frac{1}{n}|b_n| \underset{n\to\infty}{\longrightarrow} \infty. \tag{***}$$

Let us note that the analytic mechanism for constructing germs $\varphi(u) \in E'$ from the germs $(**)$ can be extracted from [3], where solutions of the differential equation of the form $(*)$ that interests us (and even of a somewhat more general form) are studied. Therefore, in the case of a principal ideal J, the construction of the corresponding subspace $E' \subset \mathfrak{F}'$ is in fact contained in [3], and moreover with fine analytic details.

Corollary of Theorem 2. *In the case when condition $(***)$ is not fulfilled, the minimal closed subspace $E' \subseteq \mathfrak{F}'$ containing the germs $(**)$ coincides with the whole space $\mathfrak{F}'$.*

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Received
2 XII 1964

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Note: Figure translations are in progress. See original paper for figures.