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Abstract

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On the Behavior of Solutions of the Equations of Hydrodynamics as the Reynolds Number Passes Through a Critical Value

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§ 1. Consider, for the Navier–Stokes equations and the equation of continuity, the following mixed problem in a bounded domain Ω:

$$Lv = \frac{\partial v}{\partial t} + \sum_{k=1}^{3} \frac{\partial v}{\partial x_k} v_k - \nu \Delta v = -\operatorname{grad} p + \mathbf{F}(x),$$

$$(\mathbf{v}(x,t) = \{v_1(x,t), v_2(x,t), v_3(x,t)\},\ x = (x_1, x_2, x_3),\ x \in \Omega), \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}|_S = 0, \qquad \mathbf{v}(x,t)\big|_{t=0} = \mathbf{a}(x).$$

With respect to the initial condition $\mathbf{a}(x)$, assume that

$$\operatorname{div} \mathbf{a}(x) = 0, \qquad \mathbf{a}|_S = 0.$$

Here S denotes the boundary of the domain Ω, and ν is a numerical parameter —the coefficient of viscosity.

We shall seek the solution of this problem in the space $\mathbf{L}_2(\Omega)$ of all vector functions $\mathbf{u}(x) = \{u_1(x), u_2(x), u_3(x)\}$, $x \in \Omega$, with components u_k from $L_2(\Omega)$. The scalar product in $\mathbf{L}_2$ is defined by the equality

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{u}, \mathbf{v})\, dx = \int_\Omega \left(\sum_{k=1}^{3} u_{kv} k \right) dx. \tag{2}$$

We reduce problem (1) to a system of ordinary differential equations in an infinite-dimensional space. To this end, following V. I. Yudovich (([1], see also [7])), we shall seek the solution of problem (1) in the form of the series

$$\mathbf{v}(x,t)=\sum_{k=1}^{\infty} z_k(t)\,\vec{\psi}_k(x), \tag{3}$$

where $\{\vec{\psi}_k(x)\}$ is a system, complete in $D(\Omega)$, of eigenvector-functions of the linear problem

$$\nu\Delta\vec{\psi}_k=-\mu_k\vec{\psi}_k+\operatorname{grad}p_k,\qquad \operatorname{div}\vec{\psi}_k=0,\qquad \vec{\psi}_k|_S=0, \tag{4}$$

with

$$\int_\Omega \psi_k^2\,dx=1.$$

The operator corresponding to this problem is self-adjoint and negative definite (see ([2])). Its spectrum $-\mu_k$ is negative, discrete, of finite multiplicity, and as $k\to\infty$, $-\mu_k\to-\infty$; $D(\Omega)$ denotes the closure in $\mathbf{L}_2$, in the metric (2), of the set of smooth, solenoidal vector functions finite in Ω.

Substituting (3) into (1) and taking the scalar product with $\vec{\psi}_s$, we obtain

$$\int_\Omega L\left(\sum_{k=1}^{\infty} z_k(t)\vec{\psi}_k(\vec{x})\right)\vec{\psi}_s\,dx=\int_\Omega \vec{F}\vec{\psi}_s\,dx\qquad (s=1,2,\ldots). \tag{5}$$

Relations (5) are equivalent to the system of ordinary differential equations

$$\dot{z}_k=-\nu\mu_k z_k+\sum_{l,m=1}^{\infty} c_{klm}z_{lz}m+F_k,\qquad k=1,2,\ldots,\infty. \tag{6}$$

The coefficients c_{klm} satisfy the relation

$$\sum_{k,l,m=1}^{\infty} c_{klm}z_{kz_{lz}}m\equiv 0. \tag{7}$$

The norm in the space z is defined as follows:

$$\|z\|=\left(\sum_{k=1}^{\infty} z_k^p\right)^{1/p}.$$

In equations (6)

$$c_{klm} = \int_\Omega (\psi_l, \nabla)\vec{\psi}_m \cdot \vec{\psi}_k \, dx.$$

The free terms in (6) are the Fourier coefficients of the body forces

$$F_k = \int_\Omega \vec{F}\vec{\psi}_k \, dx.$$

Identity (7) is the expression of the law of conservation of energy in the absence of body forces ($F = 0$) and viscosity ($\nu = 0$).

In what follows, by a solution of problem (1) we mean fulfillment of the integral relations (5). After the velocity $\mathbf{v}(x, t)$ has been obtained in the form of the series (3), the pressure p is found in the usual way (see (2)).

A summary of results on the existence and uniqueness of the solution of problem (1)–(5) is given in ([2]). In the present note new results on the study of problem (1)–(5)–(6)–(7) are set forth.

§ 2. In the works ([3–5])* we investigated the loss of stability of a stationary solution $x_0(\nu)$ of the system of ordinary equations

$$\dot{x} = f(x, \nu), \qquad x = (x_1, \dots, x_n), \qquad f(x_0(\nu), \nu) = 0.$$

Let $\lambda_1(\nu), \dots, \lambda_n(\nu)$ be the eigenvalues of the corresponding linear operator

$$A(\nu)x = (\partial f_i / \partial x_j)_{x_0(\nu)} x.$$

The stationary solution $x_0(\nu)$ loses stability as ν varies when at least one eigenvalue $\lambda(\nu)$ passes from the left half-plane $\operatorname{Re}\lambda < 0$ into the right one. The corresponding value ν is called critical, ν_{cr}: $\operatorname{Re}\lambda_1(\nu_{\text{cr}}) = 0$. We considered the case when a pair of eigenvalues crosses the imaginary axis

$$\lambda_{1,2}(\nu_{\text{cr}}) = \pm ib \neq 0, \qquad \operatorname{Re}\lambda_k < 0, \qquad k = 3, 4, \dots, n.$$

It turned out that in the "general case" the loss of stability may occur in one of two ways: a) from the stationary solution $x_0(\nu_{\text{cr}})$ that has lost stability there is born a stable limit cycle; b) the stationary solution loses stability by merging with an unstable limit cycle ([3–5])**.

* In the work ([4]) this system has the form

$$\dot{x}_k = -\nu\lambda_{kx} k + \sum_{l,m=1}^{N} A_{klm} x_{lx} m + F_k \qquad (k, l, m = 1, \dots, N).$$

** After this note had been submitted for publication, the author learned of the existence of

H. Hopf' s paper ([8]) on the birth of a cycle in a finite-dimensional space in the analytic case, possibly analogous to part of the author' s work ([3]) and the work of Yu. I. Neimark ([9]), in which the nonanalytic case is considered.

In the present note we shall show that an analogous phenomenon takes place when the laminar solution of the hydrodynamic equations (6) loses stability, if the external forces satisfy certain conditions.

§ 3. For sufficiently large viscosity ν, system (6) has a stationary solution $z(\nu)$. As ν decreases, the solution $z(\nu)$ may become unstable. The study of the loss of stability is based on the following lemma, which reduces the problem, for our purposes, to an essentially finite-dimensional one. Move the origin of coordinates z to the point $z(\nu)$: $z = z(\nu) + z'$.

Lemma 1. There exist forces $F, 0 < \|F\| < \infty$, numbers $\nu_1 > 0$, $r > 0$ such that for each number ν in the interval $\infty > \nu > \nu_1$ there exists a nonlinear invertible transformation $z' \to \tilde{z} = z' + B(z', \nu)$ of the ball $\|z'\| < r$, having the following properties:

a) $\tilde{z}_1 = z_1' - B_1(\bar{z}', \nu)^*$; $\tilde{z}_2 = z_2' - B_2(\bar{z}', \nu)$; $\bar{z} = z'$ $(\bar{z} = z_3, z_4, ...)$; $(\tilde{\bar{z}} = \tilde{z}_3, \tilde{z}_4, ...)$.

b) the operator B is completely continuous;

c) the manifold $M_\nu^{\infty-2}$ of codimension 2, defined by the equations $\tilde{z}_1 = \tilde{z}_2 = 0$, is invariant with respect to system (6)**:

$$\dot{\tilde{z}}_1 = \dot{\tilde{z}}_2 = 0 \quad \text{when} \quad \tilde{z}_1 = \tilde{z}_2 = 0.$$

Thus, if system (6) is written in the coordinates $\tilde{z}$, then the first two equations $\dot{\tilde{z}}_k$ $(k > 2)$ do not contain terms free of $\tilde{z}_1, \tilde{z}_2$. We note that the radius of the ball r does not depend on ν. With the aid of Lemma 1 one proves

Theorem 1. There exist forces $F_k, 0 < \|F\| < \infty$, and numbers $\nu_{\text{cr}} > \nu_1 > 0$ such that:

a) system (6) has a stationary solution $z(\nu)$, depending continuously on ν, when ν varies in the interval $\infty > \nu > \nu_1$;

b) the corresponding linear operator A has a pair of eigenvalues $\lambda_{1,2}(\nu)$ such that for $\nu > \nu_{\text{cr}}$, $\operatorname{Re}\lambda_{1,2}(\nu) < 0$, while for $\nu = \nu_{\text{cr}}$ these eigenvalues pass through the imaginary axis: $\lambda_{1,2}(\nu_{\text{cr}}) = \pm ib \neq 0$, $\operatorname{Re}\lambda_{1,2}(\nu) > 0$ for $\nu < \nu_{\text{cr}}$;

c) the remaining part of the spectrum of the operator A lies strictly in the left half-plane $\operatorname{Re}\lambda < -\alpha^2$, so that for $\nu > \nu_{\text{cr}}$ the equilibrium positions are stable, and for $\nu < \nu_{\text{cr}}$ they are unstable.

The character of the limiting cycle born at $\nu = \nu_{\text{cr}}$ is determined by the sign of a quantity g, depending on the coefficients of system (6) and analogous to the quantity g in works ([3],[5]). The explicit expression for g is complicated, and we shall not write it down here.

There exist forces F and numbers $\nu_{\text{cr}} > \nu_1 > 0$ such that Theorem 1 is fulfilled and either a) $g < 0$, or b) $g > 0$.

Theorem 2. There exists ν_2 ($\nu_{\text{cr}} > \nu_2 > \nu_1$) such that, for each ν ($\nu_{\text{cr}} > \nu > \nu_2$), system (6) has a stable periodic solution $\tilde{z}(t, \nu)$. In the notation of Lemma 1, $\|\tilde{z}(t,\nu)\| < O((\nu_{\text{cr}} - \nu)^{-a})$, and all solutions with initial values $\|\tilde{z}\| < r^{***}$, $\tilde{z} \notin M_\nu^{\infty-2}$, as $t \to +\infty$ tend to the periodic one.

If, however, $g > 0$, then there exists ν_3 ($\infty > \nu_3 > \nu_{\text{cr}}$) such that for ν from the interval $\nu_3 > \nu > \nu_{\text{cr}}$, system (6) has an unstable periodic solution $\tilde{z}(t, \nu)$.

The set of forces F referred to in Theorems 1 and 2 is sufficiently broad: in the coordinate system $\tilde{z}$ of Lemma 1 this set is determined by an inequality analogous to inequality (4) of work ([4]).

* It is meant that a linear transformation depending on the parameter ν has been carried out.

** The existence of an invariant manifold $M_{\nu_{\text{cr}}}^{\infty-2}$ of codimension 2 for $\nu = \nu_{\text{cr}}$ was proved by V. I. Yudovich ([6]).

*** The velocity is determined for $t > 0$.

The most difficult part of the proof of Theorem 2 is the main Lemma 2, of a nonlinear character.

Lemma 2. *Suppose that, under the conditions of Theorem 2, $g < 0$ ($g > 0$). Then, for $\nu_{\text{cr}} > \nu > \nu_2$ ($\nu_3 > \nu > \nu_{\text{cr}}$), there exists a nonlinear invertible transformation such that:*

a) $\hat{z}_1 = \tilde{z}_1, \quad \hat{z}_2 = \tilde{z}_2, \quad \hat{z}_k = \tilde{z}_k + C_k(\tilde{z}_1, \tilde{z}_2, \nu)$ $(k = 3, 4, \dots, \infty)$;

b) *the two-dimensional manifold $M_\nu^2 : \hat{z} = 0$ is invariant: $\hat{z}_k = 0$ when $\hat{z}_k = 0$ $(k = 3, 4, \dots)$;*

c) *the radius of the ball $r(\nu) > c\sqrt{|\nu_{\text{cr}} - \nu|}$.*

In the coordinates $\hat{z}$, a two-dimensional system of ordinary differential equations on the invariant manifold M_ν^2 is split off from system (6). We reduce the matrix of the linear parts of this system to the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

by a linear transformation of the plane $\hat{z}_1, \hat{z}_2$. Then system (6) on the manifold M_ν^2 takes the form of the first two equations (1_ε) from ([3],[5]). The quantity g appearing in Theorem 2 is now computed by formula (3) on p. 9 of work ([3]).

Remark 1. In an analogous way one can split off from system (6) a $2k$-dimensional system in the case when k pairs of eigenvalues pass through the imaginary axis.*

Remark 2. An invariant manifold of codimension $2k$, $M_\nu^{\infty-2k}$, analogous to the manifold $M_\nu^{\infty-2}$ of Lemma 1, also exists in the case when $2k$ eigenvalues pass through the imaginary axis.**

§ 4. In terms of hydrodynamics, Theorems 1 and 2 mean the following. For a smooth bounded domain Ω, the body forces $F(x)$ can be chosen so that:

1) *The laminar flow $v(x, \nu)$ loses stability at $\nu = \nu_{\text{cr}}$, and a pair of eigenvalues $\lambda_{1,2}$ passes through the imaginary axis;*
2) *For viscosity values ν close to ν_{cr}, in the "general case" ($g \neq 0$) there exists a periodic flow $v(x, t, \nu)$ close to $v(x, \nu)$. Moreover, either a) the periodic flow $v(x, t, \nu)$ exists** for viscosity ν somewhat less than the critical value and is stable with respect to sufficiently small perturbations (more precisely, see Theorem 2); or b) the periodic flow $v(x, t, \nu)$ exists for viscosity somewhat greater than the critical value, and it is unstable.*

In problem (1) we imposed on the boundary the no-slip condition $v|_S = 0$. By an analogous method one can also study other conditions on impermeable walls: the normal component of the velocity field on the boundary is equal to zero $(v, n)|_S = 0$, and the tangential field is prescribed.

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* With the aid of the construction presented in this note one can obtain two-dimensional tori that arise when the parameter ν passes through the "next" critical value.

** An analogous result also holds in the case of odd codimension.

*** Although all external conditions are stationary.

Note: Figure translations are in progress. See original paper for figures.