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Abstract

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MATHEMATICS

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ON THE BEHAVIOR OF A HOLOMORPHIC FUNCTION IN A NEIGHBORHOOD OF ITS ESSENTIAL SINGULAR POINT

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Consider a function $w = f(z)$, holomorphic in a neighborhood of the point $z = a$, which is an essential singular point for $f(z)$.

Definition 1. A sequence of points $\{z_n\}$, $1 > |z_n - a| > |z_{n+1} - a| > 0$, $n = 1, 2, \ldots$, $\lim\limits_{n\to\infty} |z_n - a| = 0$, will be called a sequence of $M^{(p)}$-points for the function $w = f(z)$ if:

1) $\lim\limits_{n\to\infty} (|z_n - a| - |z_{n+1} - a|)/|z_n - a| > c > 0$;

2) for any infinite subsequence $\{z_{n_\nu}\}$ of it the following holds: whatever the number ε, $0 < \varepsilon < c/2$, in the union $\bigcup_\nu D^{(p)}_{n_\nu}(\varepsilon)$ of the disks $D^{(p)}_{n_\nu}(\varepsilon)$: $|z - z_{n_\nu}| < \varepsilon|z_{n_\nu} - a|^p$, where $p \geq 1$ is natural, the function $w = f(z)$ assumes infinitely often all values from the extended complex plane, with the possible exception of two. (We note that, by condition 1), the disks $D^{(p)}_{n_\nu}(\varepsilon)$ for $0 < \varepsilon < c/2$ and sufficiently large ν do not intersect.)

Definition 2. A sequence of points $\{z_n\}$, $1 > |z_n - a| > |z_{n+1} - a| > 0$, $n = 1, 2, \ldots$, $\lim\limits_{n\to\infty} |z_n - a| = 0$, is called a sequence of $\mu^{(p)}$-points for the function $w = f(z)$, if: 1) $\lim\limits_{n\to\infty} (|z_n - a| - |z_{n+1} - a|)/|z_n - a| = c > 0$, 2) for any prescribed sequences of numbers $\{L_\nu\}$ and $\{\varepsilon_\nu\}$ such that $1 < L_1 < \cdots < L_\nu < \cdots$, $\lim\limits_{\nu\to\infty} L_\nu = \infty$, and $c/2 > \varepsilon_1 > \cdots > \varepsilon_\nu > \cdots > 0$, $\lim\limits_{\nu\to\infty} \varepsilon_\nu = 0$, one can specify a subsequence $\{z_{n_\nu}\}$ of the sequence $\{z_n\}$ such that in the disk $D^{(p)}_{n_\nu}(\varepsilon_\nu) : |z - z_{n_\nu}| < \varepsilon_\nu |z_{n_\nu} - a|^p$, $p = 1, 2, \ldots$, the function $w = f(z)$ assumes all values w from the disk $|w| < L_\nu$, with the possible exception of a set of values of diameter less than $2/L_\nu$.

Condition 2) in this definition is equivalent to the following condition:
2^1) in the disk $D^{(p)}_n(1/n) : |z - z_n| < (1/n)|z_n - a|^p$, $n = 1, 2, \ldots$, $p = 1, 2, \ldots$, the

function $w = f(z)$ assumes all values w from the disk $|w| < n$, with the possible exception of a set of values of diameter less than $2/n$.

Definition 1 can also be applied to functions meromorphic in a neighborhood of the point $z = a$, which is an essential singular point for them.

The classical Julia theorem asserts that every function meromorphic in a neighborhood of its essential singular point, which is not exceptional in the sense of Ostrovskii (in particular, every holomorphic function), possesses at least one sequence of $M^{(1)}$-points. This assertion was refined by Meiyu, who proved that every holomorphic function in a neighborhood of its essential singular point possesses a sequence of $\mu^{(1)}$-points. Lehto (^{3}) established the following fact.

Lehto' s theorem. For every function $w = f(z)$ meromorphic in a neighborhood of its essential singular point $z = a$, and not exceptional in the sense of Ostrovskii, one has

$$\overline{\lim}_{|z-a|\to 0}|z-a|\,\frac{|f'(z)|}{1+|f(z)|^2} = \infty. \tag{1}$$

Any sequence of points $\{z_n\}$ on which (1) is realized contains a subsequence that is a sequence of $M^{(1)}$-points for $w = f(z)$.

There exist, however, functions (for example, $f(z) = \exp[1/(z-a)^p]$, $p = 1, 2, ...$) for which

$$\overline{\lim}_{|z-a|\to 0}|z-a|^p\frac{|f'(z)|}{1+|f(z)|^2} = \infty, \qquad p = 1, 2, ... \tag{*}$$

For such functions the following may be stated.

Theorem 1. *Let the function $w = f(z)$ be meromorphic in a neighborhood of its essential singular point $z = a$ and satisfy condition (1^*). Then every sequence of points $\{z_n\}$ on which condition (1^*) is realized contains a subsequence that is a sequence of $M^{(p)}$-points for $w = f(z)$.*

Indeed, consider such a sequence $\{z_n\}$ and choose from it a subsequence $\{z_{n_k}\}$ so that condition 1) of Definition 1 is satisfied for it. Denote this subsequence again by $\{z_n\}$. Let $\{z_{n_\nu}\}$ be an arbitrary infinite subsequence of the sequence $\{z_n\}$. Consider the family of functions $\{F_\nu(\zeta)\}$,

$$F_\nu(\zeta) = f(z_{n_\nu} + \zeta|z_{n_\nu} - a|^p), \qquad \nu = 1, 2, \dots,$$

meromorphic in a neighborhood of the point $\zeta = 0$. We have

$$\frac{|F_\nu'(0)|}{1+|F_\nu(0)|^2} = |z_{n_\nu} - a|^p\frac{|f'(z_{n_\nu})|}{1+|f(z_{n_\nu})|^2} \to \infty \quad \text{as } \nu \to \infty.$$

Hence, by Marty' s criterion, the point $\zeta = 0$ is an irregular point in the sense of Montel for the family $\{F_\nu(\zeta)\}$, and in every disk $|\zeta| < \varepsilon$, $\varepsilon > 0$, the functions $F_\nu(\zeta)$ assume infinitely often all values of the extended complex plane, with the possible exception of two. Returning to the function $f(z)$, we obtain the assertion of Theorem 1.

In the works of Lange ([1],[2]) and in ([6]), noneuclidean analogues of the notions "sequence of $\mu^{(1)}$-points" and "sequence of $M^{(1)}$-points" were considered for functions holomorphic and meromorphic in the unit disk. The method of proof of the main assertion in ([2]) is analogous to the method used by Valiron ([5]) in his investigations of entire functions, while the main assertion in ([6]) is proved by the same method as the above Theorem 1. It was established ([6]) that every holomorphic (meromorphic) function in the unit disk that is not normal in the sense of Lehto and Virtanen ([4]) possesses a "noneuclidean sequence of $\mu^{(1)}$-points" ("noneuclidean sequence of $M^{(1)}$-points"), and such "noneuclidean sequences" necessarily contain all sequences of points of the unit disk on which the noneuclidean analogue of condition (1) is realized. The principal aim of this article is to note that analogous assertions hold also in the Euclidean case and that they can be obtained by the same methods as in ([1],[2],[6]).

Theorem 2. *Let the function $w = f(z)$ be holomorphic in a neighborhood of the point $z = a$, which is an essential singular point for $w = f(z)$, and let $\{z_n\}$ be a sequence of points converging to $z = a$ on which condition (1) is realized. Then the sequence $\{z_n\}$ contains a subsequence that is a sequence of $\mu^{(1)}$-points for $w = f(z)$. If, moreover, the function $w = f(z)$ satisfies the stronger condition (1^*) for some $p > 1$, then every sequence of points on which condition (1^*) is realized contains a subsequence that is, for $w = f(z)$, a sequence of $\mu^{(p)}$-points.*

The proof of this theorem rests on several lemmas, the first three of which are analogous to the corresponding assertions of Lange ([2]), and are proved by the same method as in ([2]) (see also ([5])).

Lemma 1. *Let the function $w = f(z)$ be holomorphic in the disk $|z - z_0| < \lambda |z_0|^p$, $p \geq 1$ a natural number, $|z_0| \neq 0$, λ a real number, and*

$f(z) \neq 0, 1$ in this disk. Then $|f(z)| < (A|f(z_0)| + A)^3$, if

$$|z - z_0| \leq \frac{1}{2\lambda} |z_0|^p,$$

where $A > 1$ is an absolute constant. If, in addition,

$$|f(z_0)| > (A')^{\lambda \eta}, \qquad 0 < \eta < \lambda,$$

where $A' > 1$ is a certain constant depending on A, then

$$|f(z)| > |f(z_0)|^{E(\eta, \lambda)},$$

when

$$|z - z_0| \leq (\lambda - \eta) |z_0|^p,$$

in which

$$E(\eta, \lambda) = \eta/2(2\lambda - \eta).$$

On the basis of Lemma 1 one proves

Lemma 2. Let the function $w = f(z)$ be holomorphic in a neighborhood of its essential singular point $z = a$. Suppose that there exist a number ε_0, $0 < \varepsilon_0 < 1$, and a sequence of points $\{z_n\}$ converging to $z = a$, such that:

1°.

$$\lim_{n\to\infty} |f(z_n)| = \infty.$$

2°. The disks

$$D_n^{(p)}(\varepsilon_0): \quad |z - z_0| < \varepsilon_0 |z_n - a|^p, \quad n = 1, 2, \dots, \quad p \geq 1,$$

do not intersect, and for each $n = 1, 2, \dots$ the disk $D_n^{(p)}(\varepsilon_0)$ contains a point ζ_n at which $f(\zeta_n) = 0$ or $f(\zeta_n) = 1$.

3°. For each $n = 1, 2, \dots$ the point ζ_n is such that

$$|z_n - \zeta_n| = \min(|z_n - z|;\ z \in D_n^{(p)}(\varepsilon_0),\ f(z) = 0 \text{ or } 1).$$

Then for any prescribed numbers L, ε, δ, $0 < L < \infty$, $0 < \varepsilon < 1$, $0 < \delta < 1$, there is a pair of points ζ, Z such that: a) the point ζ is one of the points ζ_n, $\zeta = \zeta_{n_0}$, and $|\zeta - a| < \delta$; b) the point Z lies on the line segment joining the points z_{n_0} and $\zeta_{n_0} = \zeta$, and

$$|\zeta - Z| < \varepsilon |\zeta - a|^p;$$

c) for all points z of the disk

$$|z - z_{n_0}| \leq |z_{n_0} - Z|$$

we have $|f(z)| > L$, in particular, $|f(Z)| > L$.

On the basis of Lemmas 1 and 2 one proves

Lemma 3. Let the function $w = f(z)$, the sequences of points $\{z_n\}$ and $\{\zeta_n\}$, and the number ε_0, $0 < \varepsilon_0 < 1$, be as in the hypotheses of Lemma 2. Then the sequence $\{\zeta_n\}$ contains a subsequence which is a sequence of $\mu^{(p)}$-points for $w = f(z)$.

On the basis of Lemmas 2 and 3 one proves

Lemma 4. Let the function $w = f(z)$ be holomorphic in a neighborhood of the point $z = a$, which is an essential singular point for $w = f(z)$, and let $\{z_n\}$ be some sequence of $M^{(p)}$-points $(p \geq 1)$ for the function $w = f(z)$. Consider an arbitrary sequence of numbers $\{\alpha_\nu\}$,

$$0 < \alpha_{\nu+1} < \alpha_\nu, \quad \nu = 1, 2, \dots, \quad \lim_{\nu\to\infty} \alpha_\nu = 0.$$

Then there exist two sequences of points $\{z_\nu\}$ and $\{\xi_\nu\}$ such that: a)

$$|z_\nu - \xi_\nu| < \alpha_\nu |z_\nu - a|^p, \quad \nu = 1, 2, \dots;$$

b) the sequence $\{z_\nu\}$ is a subsequence of the sequence $\{z_n\}$; c) the sequence $\{\xi_\nu\}$ is a sequence of $\mu^{(p)}$-points for the function $w = f(z)$.

Lemma 5. Let the function $w = f(z)$ be holomorphic in a neighborhood of its essential singular point $z = a$, and let the sequence $\{\zeta_n\}$ be a sequence of $\mu^{(p)}$-points $(p \geq 1)$ for the function $w = f(z)$. If a sequence of points $\{z_n\}$,

$$\lim_{n\to\infty} |z_n - a| = 0,$$

has the property that

$$|z_n - \zeta_n| < \alpha_n |z_n - a|^p, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\}$, $0 < \alpha_{n+1} < \alpha_n$, $n = 1, 2, ...$, is some numerical sequence,

$$\lim_{n\to\infty} \alpha_n = 0,$$

then $\{z_n\}$ contains a subsequence which is a sequence of $\mu^{(p)}$-points for the function $w = f(z)$.

Indeed, fix an index ν, $\nu = 1, 2, \dots$. Choose an index n_ν so that $n_\nu > \nu$ and

$$\alpha_{n_\nu} + \frac{1}{n_\nu}(1 + \alpha_{n_\nu})^p < \frac{1}{\nu},$$

and for this index n_ν consider the point z_{n_ν} from the sequence $\{z_n\}$. By the choice of the index n_ν, the disk

$$|z - z_{n_\nu}| < \frac{1}{\nu} |z_{n_\nu} - a|^p, \quad \nu = 1, 2, \dots,$$

will contain within it the disk

$$D^*_{n_\nu}: \ |z - \zeta_{n_\nu}| < \frac{1}{n_\nu} |\zeta_{n_\nu} - a|^p,$$

where ζ_{n_ν} is the point from the sequence $\{\zeta_n\}$ having index n_ν. In the disk $D^*_{n_\nu}$, by hypothesis, the function $w = f(z)$ assumes all values w from the disk $|w| < n_\nu$, with the possible exception of

sets of values of diameter less than $2/n_\nu$. Hence, in the disk

$$|z - z_{n_\nu}| < \frac{1}{\nu} |z_{n_\nu} - a|^p$$

the function $w = f(z)$ assumes all values w from the disk $|w| < n_\nu$ (and therefore also from the disk $|w| < \nu$), with the possible exception of a set of values of diameter less than $2/n_\nu < 2/\nu$. In other words, the subsequence $\{z_{n_\nu}\}$ is a sequence of $\mu^{(p)}$-points for the function $w = f(z)$.

Theorem 2 formulated above is an immediate consequence of Lemmas 4 and 5 and Theorem 1. From Lemmas 4 and 5 it follows directly:

Theorem 3. *Let the function $w = f(z)$ be holomorphic in a neighborhood of the point $z = a$, which is an essential singular point for $w = f(z)$. Then every sequence of $M^{(p)}$-points $(p \geqslant 1)$ contains a subsequence which is a sequence of $\mu^{(p)}$-points for $w = f(z)$.*

We also note the following assertion:

Theorem 4. *Let the function $w = f(z)$ be meromorphic in a neighborhood of the point $z = a$, which is an essential singular point for $w = f(z)$, and suppose that there exists a finite or infinite*

$$\lim_{n\to\infty} f(z_n) = \alpha$$

along some sequence of points $\{z_n\}$,

$$\lim_{n\to\infty} |z_n - a| = 0.$$

If there exists another sequence of points $\{z'_n\}$ such that: 1)

$$\lim_{n\to\infty} |z'_n - a| = 0;$$

2)

$$\lim_{n\to\infty} \frac{|z'_n - z_n|}{|z_n - a|^p} = 0, \qquad p \geqslant 1;$$

3) the function $f(z)$ does not have the limit α on $\{z'_n\}$, then from both sequences $\{z_n\}$ and $\{z'_n\}$ one can choose subsequences which are sequences of $M^{(p)}$-points for $w = f(z)$.

Theorem 5. *Let the function $w = f(z)$ be holomorphic in a neighborhood of the point $z = a$, which is an essential singular point for $w = f(z)$, and suppose*

$$\lim_{n\to\infty} f(z_n) = \infty$$

along some sequence of points $\{z_n\}$,

$$\lim_{n\to\infty} |z_n - a| = 0.$$

If there exists another sequence of points $\{z'_n\}$ such that: 1)

$$\lim_{n\to\infty} |z'_n - a| = 0;$$

2)

$$\overline{\lim}_{n\to\infty} \frac{|z_n - z'_n|}{|z_n - a|^p} < 1;$$

3) the function $f(z)$ is bounded on the sequence $\{z_n'\}$, then on the segments of straight lines joining the pairs of points z_n and z_n', $n = 1, 2, \ldots$, one can choose a sequence of points $\{\zeta_n\}$ which is a sequence of $\mu^{(p)}$-points for $w = f(z)$.

On the basis of Theorem 4 and Lemmas 4 and 5 it is not hard to see that the function

$$\varphi(z) = \exp\left[-\exp\left(\frac{1}{z-a}\right)\right], \qquad \arg a = 0,$$

has on the real axis $\arg z = 0$ sequences of $\mu^{(p)}$-points for every $p \geqslant 1$, while

$$\lim_{\substack{|z-a|\to 0 \\ \arg z = 0}} |z-a|^p \frac{|\varphi'(z)|}{1+|\varphi(z)|^2} = 0.$$

This example shows that condition (1^*) does not cover all sequences of $\mu^{(p)}$-points ($M^{(p)}$-points) for a holomorphic (meromorphic) function in a neighborhood of its essential singular point.

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Note: Figure translations are in progress. See original paper for figures.