V. A. EFREMOVICH, V. I. PUPKO

1965

SovietRxiv

Abstract

Full Text

V. A. EFREMOVICH, V. I. PUPKO

PROPERTIES OF EQUI-MORPHISMS AT INFINITY

(Presented by Academician L. S. Pontryagin on 20 VI 1964)

In a note by V. A. Efremovich and E. S. Tikhomirova ([1]) it was established that every equi-morphism $f : H^n \to {}'H^n$ of hyperbolic spaces induces a topological mapping $\varphi : S^{n-1} \to {}'S^{n-1}$ of their spheres at infinity. Here it will be shown that the mapping φ preserves tangency of curves. However, the order of tangency, generally speaking, is not preserved. All the more, one cannot assert that φ satisfies the Lipschitz condition (even in the case when f satisfied the Lipschitz condition).

1. We shall say that a curve $K \subset H^n$, leading to an infinitely distant point u, is tangent to the sphere at infinity at this point if its image in the **spherical Poincaré model*** (briefly: the model P) is tangent to the absolute at the point corresponding to the point u.

Theorem 1 . *Tangency of a curve with the sphere at infinity S^{n-1} is an invariant of equi-morphisms of the space H^n.*

Let us prove the following proposition, from which Theorem 1 will easily follow. A curve $K \subset H^n$, leading to a point $u \in S^{n-1}$, is tangent to S^{n-1} at this point if and only if the distance $d = d(x)$ of a point $x \in K$ from some fixed straight line directed toward u tends to infinity as $x \to u$.

1°. Let K be tangent to S^{n-1} at the point u, and let ou be the given straight line. For simplicity assume that the point o corresponds to the center of the model P. Then, denoting ox by ρ, $\operatorname{th} ox/2$ by r, and $\angle xou$ by θ, we shall have

$$\lim_{x \to u} \frac{1-r}{\theta} = 0,$$

and therefore

$$\operatorname{sh} d = \operatorname{sh} \rho \sin\theta = \frac{2r}{1+r} \frac{\sin\theta}{\theta} \frac{\theta}{1-r} \to \infty, \quad \text{i.e. } d \to \infty.$$

2°. Now let K not be tangent to S^{n-1}. Then

$$\lim_{x \to u} \frac{1-r}{\theta} \neq 0$$

(the limit may not exist at all). Therefore there exists $\varepsilon > 0$ such that, for some sequence of points $x_i \in K$ tending to u, one has

$$\frac{1 - r_i}{\theta_i} > \varepsilon.$$

In this case

$$\operatorname{sh} d_i = \operatorname{sh} \rho_i \, \sin\theta_i = \frac{2r_i}{1 + r_i} \frac{\sin\theta_i}{1 - r_i} < \frac{1}{\varepsilon}.$$

Consequently, the distances d_i remain bounded as $x_i \to u$.

Corollary. If a curve K is subjected to a bounded displacement (i.e. each of its points is moved a bounded distance), then its tangency with S^{n-1} is not destroyed.

The validity of Theorem 1 follows directly from the proposition proved above, since the image $'\Gamma$ of a rectilinear ray Γ under an equi-morphism deviates boundedly from any straight line leading to the infinitely distant point "of the ray" $'\Gamma$ (see ([1])).

---

* The use here precisely of the Poincaré model is quite essential. Thus, in the Beltrami model, tangency with the absolute would not have an invariant character (cf. Theorem 1).

** This theorem was in fact stated, albeit in an implicit form, by R. N. Khodova ([2]) (Fedorova).

Remark. If one considers the class $\mathcal{K}_{uv} = \{K\}$ of all lines $K \subset H^n$ leading to two infinitely remote points $u \neq v$ and deviating boundedly from the straight line uv, then it may be asserted that this class is invariant under equimorphisms of H^n.

2. Theorem 2. *Tangency of curves lying in S^{n-1} is an invariant of equimorphisms of H^n.*

We first prove the following proposition: two lines $K_1, K_2 \subset S^{n-1}$, issuing from a point $u \in S^{n-1}$, are tangent to one another at u if and only if on each projecting cone oK_s, $s = 1, 2$, there is a line K_s tangent to S^{n-1} at u, with K_2 deviating boundedly from K_1.

1°. Let K_1 be tangent to K_2 at the point u, and let $u_1 \in K_1$, $u_2 \in K_2$ be chosen so that the spherical distance $\beta = \widehat{u_1 u_2} = \angle u_1 o u_2$ is a small quantity of higher order relative to $\psi_1 = \widehat{u u_1} = \angle u o u_1$. Then one may take $x_s \in o u_s$ so that, as $u_s \to u$, the point x_s describes a curve K_s tangent to S^{n-1} at u, while the distance $d_{12} = x_1 x_2$ remains bounded (for example, putting $r_1 = r_2 = 1 - \beta$, we obtain

$$\operatorname{sh}\frac{d_{12}}{2} = \operatorname{sh}\rho \sin\frac{\beta}{2} = \frac{2r}{1+r}\frac{\sin\beta/2}{1-r} \to \frac{1}{2} \quad \text{as } u_s \to u.$$

Here $r = r_s = \operatorname{th} \rho_s/2,\ \rho = \rho_s = ox_s$.)

2°. Now suppose, conversely, that the lines $K_1, K_2 \subset S^{n-1}$ issuing from $u \in S^{n-1}$ are such that on oK_1, oK_2 there exist curves K_1, K_2, tangent to S^{n-1} at u and deviating boundedly from one another, i.e. there exists a correspondence between their points ($x_1 \leftrightarrow x_2,\ x_s \in K_s$) such that the distance $d_{12} = x_1x_2$ is bounded; then, applying the known formula to the triangle ox_1x_2, we obtain

$$\operatorname{sh}^2 \frac{d_{12}}{2} = \operatorname{sh}^2 \frac{\rho_2 - \rho_1}{2} + \operatorname{sh} \rho_1 \operatorname{sh} \rho_2 \sin^2 \frac{\beta}{2} \geq \operatorname{sh} \rho_1 \operatorname{sh} \rho_2 \sin^2 \frac{\beta}{2},$$

or

$$\frac{4r_1r_2}{(1+r_1)(1+r_2)} \frac{\sin^2 \beta/2}{(1-r_1)(1-r_2)} \leq \operatorname{sh}^2 \frac{d_{12}}{2}.$$

Here

$$\beta = \angle x_1 o x_2, \quad \psi_s = \angle x_{sou}, \quad \rho_s = ox_s, \quad r_s = \operatorname{th} \rho_s/2, \quad s = 1, 2;$$

therefore $\beta^2 \leq \text{const} \cdot (1 - r_1)(1 - r_2)$, or $\beta^2 = o(\psi_1)o(\psi_2)$ as $x_s \to u$. Since, moreover, $|\psi_2 - \psi_1| \leq \beta$, i.e. $(\psi_2 - \psi_1)^2 \leq \text{const} \cdot (1 - r_1)(1 - r_2)$, and $1 - r_s = o(\psi_s)$, it follows that ψ_1 and ψ_2 are equivalent infinitesimals, and hence $\beta = o(\psi_s)$, i.e. the curves K_1, K_2 are tangent.

Proof of Theorem 2. Let K_1 be tangent to K_2. Then their images K_1', K_2' must also be tangent to one another, and this is why: the curves $K_s \subset oK_s,\ s = 1, 2$, chosen as indicated above (in 1°), pass into curves K_s', also tangent to the absolute (Theorem 1) and deviating boundedly from one another. But, unfortunately, they need not lie on the cones $o'K_s'$. Projecting orthogonally each point $x_s' \in K_s'$ onto the corresponding generator of the cone $o'K_s'$, we obtain new curves $K_s'' \subset o'K_s'$, also deviating boundedly from one another* and, by virtue of the corollary, tangent to S^{n-1}. Therefore K_1' is tangent to K_2'.

3. Theorem 3. *The order of tangency with the absolute of the model P and the order of tangency of curves on the absolute are not invariant under equimorphisms of H^n.*

This theorem is proved by constructing an example of a mapping $F : H^n \to H'^n$ satisfying a two-sided Lipschitz condition on H^n and not preserving the order of tangency of curves at a certain point $u \in S^{n-1}$ (thereby a stronger assertion will be proved, namely: orders of tangency are not invariant even under L-isomorphisms, i.e. under mappings satisfying a two-sided Lipschitz condition).

* The image of a rectilinear ray deviates from rectilinearity by an amount bounded in the aggregate of all rays issuing from the fixed point o (see (1)).

The mapping F for $n=2$. Let ρ, θ, $\rho \geq 0$, $0 \leq |\theta| \leq \pi$, be polar coordinates on H^2. Divide H^2 into four (closed) regions and define $F : (\rho, \theta) \to ({}'\rho, {}'\theta)$ in each of them by the formulas:

$$\begin{array}{lll}
\text{I region.} & \rho \leq \operatorname{ar\,sh} 1; & {}'\rho = 2\rho, \quad {}'\theta = \theta. \\
\text{II} & \operatorname{sh}\rho \geq 1,\ |\theta| \operatorname{sh}\rho \leq 1; & {}'\rho = 2\rho, \quad {}'\theta = \theta / \operatorname{sh}\rho. \\
\text{III} & \dfrac{1}{\operatorname{sh}\rho} \leq |\theta| \leq 1; & {}'\rho = \rho + \operatorname{ar\,sh}\dfrac{1}{|\theta|}, \quad {}'\theta = \theta' \operatorname{sgn}\theta. \\
\text{IV} & \operatorname{sh}\rho \geq 1,\ 1 \leq |\theta| \leq \pi; & {}'\rho = \rho + \operatorname{ar\,sh} 1, \quad {}'\theta = \theta.
\end{array}$$

It is easy to see that F is single-valued and continuous everywhere. Let us check that F satisfies the two-sided Lipschitz condition in each region. For regions I and IV this is obvious. It remains to check it for regions II and III. It is sufficient to verify the Lipschitz condition in differential form and only along each of the coordinate lines.

A direct calculation shows that in region II, for $d\rho = 0$, the ratio of the length elements $d's/ds \leq 2\sqrt{2}$, and for $d\theta = 0$, $d's/ds \leq 2\sqrt{5}$; in region III, for $d\rho = 0$, $d's/ds \leq \sqrt{33}$, and for $d\theta = 0$, $ds' = ds$.

In the same way the Lipschitz condition is also verified for the inverse mapping. Thus F satisfies the two-sided Lipschitz condition.

Now consider the curve $\operatorname{th}\rho/2 = 1 - \theta^{m+1}$, or $r = 1 - \theta^{m+1}$, where $r = \operatorname{th}\rho/2$ and θ are polar coordinates on the model P. This curve is tangent to the absolute at the point $(1, 0)$, and the order of tangency is m. For sufficiently small θ the corresponding part of the curve lies entirely in region III. Applying to it our mapping $F : (r, \theta) \to ({}'r, {}'\theta)$, we obtain:

$$1 - {}'r = \frac{\theta^{m+1}(\theta + \sqrt{1+\theta^2} - 1)}{\theta + \sqrt{1+\theta^2} - \theta^{m+1} + 1};$$

as $\theta \to 0$, $1 - {}'r \sim \frac{1}{2}\theta^{m+2} = \frac{1}{2}{}'\theta^{m/2+1}$, i.e., the order ${}'m$ of tangency of the transformed curve has become ${}'m = m/2$. Hence ${}'m = m$ only when $m = 0$, which is in agreement with Theorem 1.

The example constructed is directly carried over to the n-dimensional case. Thus, for $n = 3$, in spherical coordinates ρ, θ, ψ (θ is the "zenith distance," ψ the "azimuth") we define the mapping $F : H^3 \to {}'H^3$ by the same formulas, adjoining to them ${}'\psi = \psi$. It is easy to see that the order of tangency at the zenith of curves lying on the absolute of the model P changes. For example, the curves $\psi = 0$ and $\psi = \theta^2$, lying on the absolute, pass into ${}'\psi = 0$ and ${}'\psi = {}'\theta$.

Remark. Our mapping ${}'F$ induces on the absolute a mapping Φ that does not satisfy the Lipschitz condition, and therefore inside the model P the Lipschitz condition is also not fulfilled. Thus, an L-isomorphism of Lobachevsky space need not be represented by an L-isomorphism of the model P. The example of the mapping F refutes a very natural hypothesis, which seemed all the more probable since the following assertion is true:

If an equimorphism $f : E^n \to {}'E^n$ of Euclidean spaces induces a topological mapping $\varphi : S^{n-1} \to {}'S^{n-1}$ of their infinitely distant spheres, and the deviation of the image T of any ray $\Gamma = ou,\ u \in S^{n-1}$, from the ray ${}'o'u$ is bounded over the totality of all rays Γ, then φ satisfies the two-sided Lipschitz condition. The example of the mapping F, and consequently also Theorem 3, are entirely due to V. I. Pulko.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
9 VI 1964

REFERENCES

1. V. A. Efremovich, E. S. Tikhomirova, *DAN*, **152**, No. 5 (1962).

2. R. N. Khodova, *Scientific Notes of the Ivanovo Pedagogical Institute*, **10** (1956).

Note: Figure translations are in progress. See original paper for figures.