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Abstract

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M. I. KRATKO

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As is known, in order to make precise the intuitive notion of an algorithm, E. Post used special calculi ([1]). It turned out, however, that the study of such calculi is of interest not only from the point of view of the theory of algorithms. Post calculi have much in common with the formal languages studied by N. Chomsky, A. V. Gladkii, and others. A connection has also been established between Post calculi and finite automata ([2]). Recently a number of works devoted to such calculi have appeared ([2–6]).

In the present note we consider one class of Post calculi, for which one conjecture of D. Buchi is proved. In §1 we recall the basic notions. In §2 some known results are given. The new results are contained in §3.

1°. In specifying a Post calculus, two alphabets are fixed: a principal alphabet $\mathfrak{A}$ and an auxiliary alphabet $\mathfrak{A}'$. A Post calculus $[\mathfrak{C}, \Sigma]$ consists of: 1) a finite set of words $\mathfrak{C}$ in the alphabet $\mathfrak{A} \cup \mathfrak{A}'$, called axioms, and 2) a finite collection Σ of rules of inference, called productions.

In what follows we shall use productions of the following three types:

$$\text{I.} \quad A_1X,\ A_2X, \dots, A_{kX} \to BX \qquad (\text{Ia} \quad AX \to BX).$$

$$\text{II.} \quad XD_1,\ XD_2, \dots, XD_m \to XE \qquad (\text{IIa} \quad XD \to XE).$$

$$\text{III.} \quad F_1X,\ F_2X, \dots, F_{nX} \to XG \qquad (\text{IIIa} \quad FX \to XG).$$

Here $A, A_1, \dots, A_k,\ D, D_1, \dots, D_m,\ F, F_1, \dots, F_n,\ B,\ E,\ G$ are certain words in the alphabet $\mathfrak{A} \cup \mathfrak{A}'$; X is a letter not belonging to the alphabet $\mathfrak{A} \cup \mathfrak{A}'$. A production of type I directly derives the word S from the set of words $\{R_1, R_2, \dots, R_k\}$ if there exists a word X in the alphabet $\mathfrak{A} \cup \mathfrak{A}'$ such that $R_1 = A_1X,\ R_2 = A_2X, \dots,\ R_k = A_{kX}$ and $S = BX$. In a similar way the notion of direct derivability is defined for productions of types II and III.

We say that a word S is derivable in the Post calculus $[\mathfrak{C}, \Sigma]$ if: 1) it is a word in the principal alphabet $\mathfrak{A}$; 2) there exists such a sequence of words

$$T_1,\ T_2, \dots,\ T_p, \tag{*}$$

that $T_p = S$ and every T_i is either an axiom or is directly derived from some set of words occurring in the sequence $(*)$ earlier than T_i, by means of one of the productions $\sigma \in \Sigma$.

The set of derivable words in $[\mathfrak{C}, \Sigma]$ will be denoted by $\tau(\mathfrak{C}, \Sigma)$. Post calculi $[\mathfrak{C}, \Sigma]$ and $[\mathfrak{C}', \Sigma']$ are called equivalent if

$$\tau(\mathfrak{C}, \Sigma) = \tau(\mathfrak{C}', \Sigma').$$

2°. A normal Post calculus contains a single axiom, and all its productions have the form IIIa.

It is known that for every recursively enumerable set $\mathfrak{M}$ there exists a normal Post calculus $[\mathfrak{C}, \Sigma]$ such that $\tau(\mathfrak{C}, \Sigma) = \mathfrak{M}$.

In the paper (1) E. Post noted that calculi in which productions of the form

$$A_1X,\ A_2X \to BX$$

$$XD \to XE \quad \text{or} \quad XD_1,\ XD_2 \to XE$$

are used generate only recursive sets of words. Subsequently J. Büchi showed that:

1) Post calculi with productions of type Ia generate only regular (in the sense of Kleene (7)) sets of words. For each such calculus $[\mathfrak{S}, \Sigma]$ one can effectively construct a finite automaton that recognizes exactly the set of words $\tau(\mathfrak{S}, \Sigma)$.

2) For each finite automaton recognizing some set of words $\mathfrak{X}$, one can effectively construct a Post calculus $[\mathfrak{S}, \Sigma]$ whose auxiliary alphabet is empty, all productions have the form

$$AX \to ABX$$

(a special case of Ia), and $\tau(\mathfrak{S}, \Sigma) = \mathfrak{X}$.

J. Büchi posed the problem of investigating Post calculi in which productions of types I and IIa are used.* He conjectured that such calculi also generate only regular sets of words (2). The interest of such calculi becomes clearer if one recalls E. Post' s remark (1) that for every normal calculus $[\mathfrak{S}, \Sigma]$ there exists an equivalent calculus $[\mathfrak{S}', \Sigma']$ with productions of the form

$$A_1X,\ A_2X \to BX,$$

$$XD_1,\ XD_2 \to XE,$$

i.e., with productions of types I and II.

3°. **Theorem.** *Post calculi with productions of types I and IIa generate only regular sets of words. For each such calculus* $[\mathfrak{S}, \Sigma]$ *one can effectively construct a finite automaton that recognizes exactly the set of words* $\tau(\mathfrak{S}, \Sigma)$.

It follows from this that, if all Post calculi with productions of types I–III are divided into classes depending on the form of the productions, then each such class generates either all recursively enumerable sets of words** or only regular ones.

Post calculi can be generalized by admitting infinite sets of words as the set of axioms. Naturally, in this case only those calculi are of interest whose sets of axioms are given effectively.

The assertion of the theorem remains true also for such calculi, provided only that the set of axioms is a regular set. If, however, arbitrary primitive recursive sets (or even rudimentary sets) are allowed as sets of axioms, then, as is easy to see, already the class of calculi with productions of type Ia generates all recursively enumerable sets of words.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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CITED LITERATURE

1. E. Post, *Am. J. Math.*, **65**, 2 (1943).
2. J. R. Büchi, *Arch. Math. Logik u. Grundlagenforsch.*, **6**, 3–4 (1964).
3. Wang Hao, *Math. Ann.*, **152**, 1 (1963).
4. Watanabe Shigeru, *Proc. Symp. Math. Theory Automata*, N. Y., 1963.
5. S. Maslov, DAN, **147**, No. 4 (1962).
6. S. Maslov, DAN, **152**, No. 2 (1963).
7. S. K. Kleene, *Collected volume Automata*, IL, 1956.

* An analogous problem can be formulated for Post calculi with productions of types Ia and II. However, it is easy to note that for every Post calculus

with productions of types I and IIa there exists an equivalent calculus with productions of types Ia and II, and conversely.

** In some fixed alphabet.

Note: Figure translations are in progress. See original paper for figures.