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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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LOCAL STRUCTURE OF DARBOUX SURFACES

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MATHEMATICS

1. For brevity, in what follows, instead of *infinitesimal bending* we shall simply write *bending*.
2. Let a surface R, with radius vector $\mathbf{r} = \mathbf{r}(u, v)$, be subjected to a bending. This means that at each point of the surface R there is defined a vector $\mathbf{z}(u, v)$ such that

$$(d\mathbf{r}\, d\mathbf{z}) = 0 \tag{1}$$

(for more details see ([3],[4]); in the names of fields associated with the bending field $\mathbf{z}$, we follow the terminology of ([4])).

Darboux ([1]) considered, in addition to the given surface R itself, 11 more surfaces associated with the given bending field $\mathbf{z}(u, v)$ and the surface R. Our note is devoted to the study "in the small" of the extrinsic-geometric structure of Darboux surfaces when the bent surface R has positive curvature and belongs to the class $C^{n,\alpha}$, $\alpha > 0$, $n \geq 2$. The works ([2],[3]) are close to our topic. In ([2]) Libman proved the absence of supporting planes for the rotation diagram when R is an analytic surface. We establish this result for $R \in C^{2,\alpha}$ (see the corollary to Theorem 2). Cohn-Vossen ([3]) showed the isolatedness of the stationary points of the rotation diagram for $R \in C^3$, relying on Carleman' s theorem on the isolatedness of zeros of a solution of a generalized Cauchy–Riemann system. In the same work Cohn-Vossen raised the question of the possibility of such a proof of this fact that would reveal its topological-geometric meaning. The answer to this question is contained in Theorem 1, from which it follows that the isolatedness of stationary points follows from the topological properties of the interior (in the sense of Stoilow ([5])) mappings.

Fig. 1

3. As is known, from a given bending field $\mathbf{z}(u,v)$ one uniquely determines the rotation field $\mathbf{y}(u,v)$ (by the relation $d\mathbf{z} = [\mathbf{y}\, d\mathbf{r}]$) and the displacement field $\mathbf{s}(u,v) = \mathbf{z} - [\mathbf{y}\mathbf{r}]$. Considering $\mathbf{z}$, $\mathbf{y}$, and $\mathbf{s}$ as radius vectors drawn from the origin of coordinates, we obtain respectively the surfaces Z (the bending diagram), Y (the rotation diagram), and S (the displacement diagram). Thus, a pair of vectors $(\mathbf{r}, \mathbf{z})$, connected by relation (1), generates four surfaces R, Z, Y, and S, and two new pairs–$(\mathbf{z}, \mathbf{r})$ and $(\mathbf{s}, \mathbf{y})$–from which one may continue analogous constructions and obtain the remaining Darboux surfaces (see the scheme (Fig. 1); arrow 1 indicates the bending diagram, arrow 2 indicates the rotation diagram).

4. Since the consideration is local, we may assume that the surface R is given by the equation $z = f(x,y)$ over the domain $D : x^2 + y^2 < \varepsilon^2$. Let $p = \partial f/\partial x,\ q = \partial f/\partial y,\ r = \partial^2 f/\partial x^2,\ s = \partial^2 f/\partial x\, \partial y,\ t = \partial^2 f/\partial y^2,\ rt - s^2 > 0$ in D. Let the bending vector be $\mathbf{z} = (\xi, \eta, \zeta)$; then the rotation vector is $\mathbf{y} = (\zeta_y, -\zeta_x, \psi)$, $\psi = \eta_x + p\zeta_y = -\xi_y - q\zeta_x$. For the component $\zeta(x,y)$ we have the equation

$$t\zeta_{xx} - 2s\zeta_{xy} + r\zeta_{yy} = 0.$$

Since $rt - s^2 > 0$, it follows that $\Delta = \zeta_{xx}\zeta_{yy} - \zeta_{xy}^2 \le 0$. Consequently, the surface $z = \zeta(x,y)$ does not permit caps to be cut off (in the terminology of ([6]), p. 46).

The choice of the coordinate system $(Oxyz)$ can be specialized. Namely, the following is true.

Lemma 1. *Let a point $M_0 \in R$. It is asserted that the axes Ox, Oy, Oz can be chosen so that the surface R in a neighborhood of the point M_0 is represented by the equation $z = f(x,y)$, and so that to this point M_0 there corresponds on the plane (x,y) the point $M(0,0)$, at which $p \neq 0,\ q \neq 0,\ p + q = 0,\ r = t,\ s = 0$.*

In proving all the theorems given below it is convenient to assume that the axes Ox, Oy, Oz have been chosen so that, for the point $M_0 \in R$ under consideration, the assertions of Lemma 1 are satisfied.

5. The main points in the study of the local structure of Darboux surfaces are based on the following lemmas.

Lemma 2. *Let the coordinate system $(Oxyz)$ be chosen as in Lemma 1, and let at the point $M(0,0)$ we have $\xi_x = \xi_y = 0$. Then there exist constants $c_1, c_2,\ c_1^2 + c_2^2 \neq 0$, and an integer $n \ge 1$, such that in a neighborhood of the point $M(0,0)$ the relations*

$$\xi_x(x,y) = \rho^n(c_1 \cos n\theta + c_2 \sin n\theta) + o(\rho^{n+\alpha'}),$$

$$\xi_y(x,y) = \rho^n(c_2 \cos n\theta - c_1 \sin n\theta) + o(\rho^{n+\alpha'}),$$

hold, where $\alpha' > 0,\ \rho = \sqrt{x^2 + y^2},\ x = \rho\cos\theta,\ y = \rho\sin\theta.$

Lemma 3. *Let a smooth surface* $X : z = \varphi(x, y)$ *not permit caps to be cut off, and suppose that in a neighborhood of some point* $\bar{M}$ *of it there is no continuum,* passing into a point under the spherical mapping of the surface X. *Then the point* $\bar{M}$ *has on* X *a canonical neighborhood* $U(\bar{M}) \subset X$ *(i.e. a neighborhood at each point of which the normal to* X *is not parallel to the normal to* X *at the point* $\bar{M}$*).*

Corollary. *If, under the spherical mapping of a smooth surface from which no cap can be cut off, no continuum is mapped to a point, then this mapping is interior.*

6. **Local structure of the surface** Y**.** Since

$$[\mathbf{y}_x \mathbf{y}_y] = (\zeta_{xx}\zeta_{yy} - \zeta_{xy}^2)[\mathbf{r}_x \mathbf{r}_y] = \Delta[\mathbf{r}_x \mathbf{r}_y],$$

it follows that Y is a surface of negative curvature, with the only possible singularities at stationary points, i.e. at points at which $\Delta = 0$.

Theorem 1. *The mapping* $T : u = \zeta_x,\ v = \zeta_y$ *is interior in the sense of Stoilow, and its branch points correspond to stationary and only stationary points.*

Corollary 1. *The stationary points of the surface* Y *are situated in isolation.*

The proof follows from Theorem 1 and from Stoilow' s theorem on the isolation of branch points of an interior mapping; see ([5]).

Corollary 2. *The surface* Y *at each point has a normal or a unique limiting position of the normal.*

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* By a continuum we mean a connected closed set.

Corollary 3. The stationary points are branch points of the surface Y*.

Theorem 2. Let at the point $M(0, 0)$ the order of the saddle-like character (see the definition in (7)) of the surface $z = \zeta(x, y)$ be equal to m. Then, in a neighborhood of the point M, the intersection of the surface Y with its tangent plane at the point M consists of $2m$ branches. The intersection with any other plane consists of $2m - 2$ branches.

Remark 1. If the point M is stationary, then by the tangent plane to Y at the point M we mean the limiting position at M of the planes tangent to Y, which exists and is unique by virtue of Corollary 2 of Theorem 1.

Remark 2. Theorem 2 points to the "nonstandard" character of the branching of the surface Y at the stationary point M. Usually, at saddle branch points the number of branches of the intersection of the tangent plane with the surface is exactly twice as large as the number of branches of the intersection of the surface with any other plane. But for the surface Y this regularity is violated when $m \geqslant 3$.

It can be shown that for the surface $z = \zeta(x, y)$ the order of saddle-like character at any point is not less than two. Therefore Theorem 2 implies

Corollary. The surface Y has no support plane at any of its points (even locally).

7. Local structure of the surface H. It can be shown that the radius vector $\mathbf{h}$ of the surface H is expressed by the formula

$$\mathbf{h} = -\mathbf{n}/(\mathbf{rn}), \tag{2}$$

where $\mathbf{n}$ is the unit normal to R. We note that the surface H is the only one of the 11 Darboux surfaces that depends only on the initial surface R and does not depend on the bend z under consideration.

Theorem 3. Let, in a neighborhood of the point under consideration $M_0 \in R$, the surface R be visible from one side from the origin of coordinates. Then in the corresponding neighborhood the surface H has positive curvature and belongs to the same class of regularity as the surface R.

Remark. Starting from (2), one can reconstruct the surface R, if its support function $A(\mathbf{n})$ is given as a function of the normal $\mathbf{n}$ to R, i.e., solve the Minkowski problem. The required radius vector $\mathbf{r}$ is given by the formula

$$\mathbf{r} = \frac{[\mathbf{h}_u \mathbf{h}_v]\, A(\mathbf{n})}{(\mathbf{h}_u \mathbf{h}_v \mathbf{n})}.$$

8. As is seen from the scheme (Fig. 1), the Darboux surfaces are grouped in pairs as follows: (R, H), (Y, S), (Z, P), (G, Σ), (F, W), (Q, Ω), so that in one pair fall those surfaces which have identical combinatorial relations, respectively, with R and H. In each pair it is enough to study the structure of one surface, say the one referring to R. Then the second surface, referring to H, by virtue of Theorem 3, will have exactly the same local structure as the first. More precisely: suppose that for the surface referring to R some local property has been established, described only in terms of the positivity of the curvature of the surface R and in terms of its class of regularity. Then this same property is also true for the surface referring to H, if the conditions of Theorem 3 are fulfilled.

9. Local structure of the surface S. On the basis of what was said in §8, we may assert that, when the conditions

* We call a point M of a surface X a **branch point** of the surface X, if there is some plane Π and an arbitrarily small neighborhood $U \subset X$ of the point M, such that, under projection of the neighborhood U onto the plane Π, the

following conditions are satisfied: a) the point M is projected to some point $M' \in \Pi$, for which the point M is the unique preimage in U; b) the projection of U fills in on Π some neighborhood U' of the point M'; c) each point $M'' \in U'$, $M'' \neq M'$, has n preimages in U, $n \geqslant 2$, and the number n is the same for all points of $U' \setminus M'$.

by Theorem 3, the local structure of the surface S coincides with the already studied structure of Y. If the conditions of Theorem 3 are violated, i.e., if a line $(\mathbf{rn}) = 0$ passes through the point $M_0 \in R$ under consideration, then to this line on the surface S there corresponds a special line along which the regularity of the surface S is violated. But it can be shown that in any case the surface S nowhere has supporting planes.

10. Local structure of the surfaces Z and P. Let $\mathbf{N} = (-p, -q, 1)$ be a vector parallel to the normal to R. Introduce the function $V \equiv (\mathbf{yN}) = \eta_x + q\zeta_x$. The level line $V = 0$ will be called a V-line.

Theorem 4. *The surface Z has nonpositive curvature with possible isolated zeros. It is regular everywhere, except for the V-line, to whose simple arcs on Z there correspond smooth cuspidal edges. At each point of these edges (with the exception of a finite number of points of their intersections) the tangent plane to Z has a definite limiting position, and the limiting plane is not supporting for Z. All other planes passing through the tangent to the edge are locally supporting for Z (i.e., in a neighborhood of the edges the surface Z has a structure similar to that of a pseudosphere). At the points of intersection of the edges the surface Z has no supporting plane.*

The surface P has an analogous structure; its edges correspond to the line $(\mathbf{rz}) = 0$. If on the surface R there is a line $(\mathbf{rn}) = 0$, then as one approaches the points of this line the corresponding points on P go off to infinity.

11. Local structure of the surfaces W and F.
Theorem 5. *The surface W is of nonpositive curvature; it has no self-intersections anywhere and is regular everywhere, except for the V-line, on which its radius vector tends to infinity.*

The surface F has an analogous structure at points where $(\mathbf{rn}) \neq 0$. If on the surface R there is a line $(\mathbf{rn}) = 0$, then to this line on F there corresponds a smooth cuspidal edge, along which the surface F has the same local structure as the surface Z along its edge. At the points of the line $(\mathbf{rz}) = 0$ the surface F goes off to infinity.

12. Without going into details concerning the structure of the surfaces Σ, G, Q, and Ω, we note only that Σ and G have nonnegative curvature, while Ω and Q have nonpositive curvature.

In conclusion, I take this opportunity to express my sincere gratitude to my advisor N. V. Efimov for his constant attention and interest in this work.

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Note: Figure translations are in progress. See original paper for figures.