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Abstract

Full Text

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DECOMPOSITION OF A REPRESENTATION OF A REDUCTIVE LIE ALGEBRA INTO REPRESENTATIONS OF ITS REGULAR REDUCTIVE SUBALGEBRAS OF MAXIMAL RANK

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MATHEMATICS

1. Introduction.

Let g be a complex reductive algebra of rank l, g' its complex reductive regular subalgebra of the same rank,* and h a Cartan subalgebra of g and g'. Let Δ_+ be the set of positive roots φ_k of the algebra g ($k = 1, 2, \dots, (r-l)/2$; r is the order of the algebra g), and $\Delta'_+ \subset \Delta_+$ the set of positive roots φ_{k_i} of the subalgebra g' ($i = 1, 2, \dots, (r'-l')/2$; r' is the order of the subalgebra g''). Let, by definition, $\mu \in I$ if

$$2(\mu, \varphi_k)/(\varphi_k, \varphi_k) = \text{an integer for all } \varphi_k \in \Delta_+. \tag{1}$$

Similarly, let $\mu' \in I'$ if

$$2(\mu', \varphi_{k_i})/(\varphi_{k_i}, \varphi_{k_i}) = \text{an integer for all } \varphi_{k_i} \in \Delta'_+. \tag{2}$$

Since $\Delta'_+ \subset \Delta_+$, condition (2) is weaker than (1), i.e. $I \subset I'$. Recall that the set $I(I')$ is the set of all weights of all representations of the algebra $g(g')$. Let $\pi_\lambda(\pi'_{\lambda'})$ be an irreducible representation of the algebra g (of the subalgebra g') with highest weight λ (λ') in a finite-dimensional vector space $V_\lambda(V'_{\lambda'})$, and let $m_\lambda(m'_{\lambda'})$ be a function on $I(I')$ assigning to each vector $\nu \in I$ ($\nu' \in I'$) the multiplicity $m_\lambda(\nu)$ ($m'_{\lambda'}(\nu')$) of its occurrence as a weight in the representation $\pi_\lambda(\pi'_{\lambda'})$.

The aim of the present paper is to obtain a formula showing with what multiplicity $\rho_\lambda(\lambda')$ the irreducible representation $\pi'_{\lambda'}$ of the subalgebra g' occurs in the irreducible representation π_λ of the algebra g.

2. Formula for the multiplicity of an irreducible representation of the subalgebra.

We shall rely on two well-known general theorems of character theory:

$$\chi_\lambda(x) = \sum_{\nu \in I} m_\lambda(\nu) \exp(\nu, x); \tag{3}$$

$$\chi_\lambda(x) = \sum_{\lambda' \in I'_D} \rho_\lambda(\lambda') \chi'_{\lambda'}(x), \tag{4}$$

where (ν, x) is the Killing–Cartan bilinear form on g; $x \in h$; $\chi_\lambda(x)$, $\chi'_{\lambda'}(x)$ are the characters of the representations $\pi_\lambda, \pi'_{\lambda'}$, respectively, and $\overline{I'_D}$ is the set of all highest weights of $\pi'_{\lambda'}$ (D' is a certain once and for all fixed Weyl chamber of the subalgebra g').**

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* By definition, a subalgebra g is called a **reductive regular subalgebra of rank** l if it is the direct sum of a regular semisimple subalgebra of rank l' and a commutative algebra of rank $l - l'$. (A semisimple subalgebra g'' of the algebra g is called **regular** if its root system is part of the root system of the algebra g.)

** For more details on the definition of I'_D, see ([1]). We note that by the Weyl group of the reductive algebra G one means the Weyl group of its semisimple part G', extended identically to the Abelian complement to the Cartan subalgebra h' of the algebra G'.

Substituting (3) into (4), we have:

$$\sum_{\nu \in I} m_\lambda(\nu) \exp(\nu, x) = \sum_{\lambda' \in D'} \rho_\lambda(\lambda') \sum_{\nu' \in I'} m'_{\lambda'}(\nu') \exp(\nu', x). \tag{5}$$

Introduce the function $\delta_{\nu I}$, equal to one for $\nu \in I$ and equal to zero for $\nu \notin I$. Then, rewriting (5) in the form

$$\sum_{\nu \in I'} m_\lambda(\nu) \delta_{\nu I} \exp(\nu, x) = \sum_{\nu' \in I'} \sum_{\lambda' \in I_{D'}} \rho_\lambda(\lambda') m'_{\lambda'}(\nu') \exp(\nu', x) \tag{6}$$

and equating the coefficients of equal exponents, we obtain, for $\nu \in I'$,

$$m_\lambda(\nu) \delta_{\nu I} = \sum_{\lambda' \in I_{D'}} \rho_\lambda(\lambda') m'_{\lambda'}(\nu). \tag{7}$$

Our aim is to find an explicit form of $\rho_\lambda(\lambda')$. To this end we shall use Lemma 2.4 of [2], which can be formulated as follows.

Let $P(\mu)$ be the function of partitioning the vector μ into a sum of positive roots $\varphi_1, \varphi_2, \dots, \varphi_{k-1}, \varphi_k, \varphi_{k+1}, \dots, \varphi_n$, $n = (r - l)/2$. Then the first-order difference function

$$P^{(\varphi_k)}(\mu) = P(\mu) - P(\mu - \varphi_k) \tag{8}$$

is the function of partitioning the vector μ into a sum of positive roots $\varphi_1, \varphi_2, \dots, \varphi_{k-1}, \varphi_{k+1}, \dots, \varphi_n$. We note that the multiplicity of the weight ν of the representation π_λ of the algebra g is expressed in terms of $P(\mu)$ as follows:

$$m_\lambda(\nu) = \sum_{\sigma \in W} \operatorname{sgn} \sigma \, P[\sigma(\beta + \lambda) - (\beta + \nu)], \tag{9}$$

which is a trivial generalization of Kostant' s formula [1, 3] to reductive algebras. Here W is the Weyl group of the algebra g and

$$\beta = \frac{1}{2} \sum_1^n \varphi_k.$$

It follows directly from Lemma 2.4 of [2] that the difference function of order $n' = (r' - l')/2$

$$P^{(\Delta'_+)}(\mu) \equiv P^{(\varphi_{k_1}, \varphi_{k_2}, \dots, \varphi_{k_{n'}})} = \sum_{j_1=0}^{1} \cdots \sum_{j_{n'}=0}^{1} (-1)^{\sum_1^{n'} j_i} P(\mu - j_i \varphi_{k_i}) \tag{10}$$

is the function of partitioning the vector μ into a sum of positive roots of the algebra g that do not belong to the subalgebra g'. We also note that the difference function of order n

$$P^{(\Delta_+)}(\mu) \equiv P^{(\varphi_1, \varphi_2, \dots, \varphi_n)}(\mu) = \delta_{\mu 0}, \tag{11}$$

since, by Lemma 2.4, $P^{(\Delta_+)}(\mu)$ is the function of partitioning the vector μ into a sum of zero roots.

Form the sum

$$m_\lambda^{(\Delta'_+)}(\nu) \equiv \sum_{j_1=0}^{1} \cdots \sum_{j_{n'}=0}^{1} (-1)^{\sum_1^{n'} j_i} m_\lambda(\nu + j_i \varphi_{k_i}). \tag{12}$$

Substituting equality (7) into (12) and using formula (9) and definition (10), we obtain

$$\delta_{\nu I}\sum_{\sigma\in W}\operatorname{sgn}\sigma P^{(\Delta'_+)}[\sigma(\beta+\lambda)-(\beta+\nu)]=\sum_{\lambda'\in I_{D'}}\rho_\lambda(\lambda')\sum_{\sigma'\in W'}\operatorname{sgn}\sigma' P'^{(\Delta'_+)}[\sigma'(\beta'+\lambda')-(\beta'+\nu)], \tag{13}$$

where $\nu\in I'$ and $P'^{(\Delta'_+)}(\mu)$ is composed according to the prescription (10) from the functions of decomposition $P'(\mu)$ of the vector μ into the sum of positive roots φ_{k_i} of the subalgebra g'; W' is the Weyl group of the subalgebra g', and

$$\beta'=\frac{1}{2}\sum_{i=1}^{n'}\varphi_{k_i}.$$

Taking (11) into account, we have $P'^{(\Delta'_+)}(\mu)=\delta_{\mu 0}$, i.e. (13) can be rewritten in the form

$$\delta_{\nu I}\sum_{\sigma\in W}\operatorname{sgn}\sigma P^{(\Delta'_+)}[\sigma(\beta+\lambda)-(\beta+\nu)]=\sum_{\sigma'\in W'}\operatorname{sgn}\sigma'\rho_\lambda[\sigma'^{-1}(\beta'+\nu)-\beta'], \tag{14}$$

where $\nu\in I'$, and the summation on the right-hand side of (14) is taken over all σ' such that $\sigma'^{-1}(\beta'+\nu)-\beta'\in I'_{D'}$.

It is easy to see that there exists only one σ' satisfying this condition, namely the Weyl reflection which takes the region $I'_{D'}+\beta'$ of the set I' into the region I' containing the vector $\nu+\beta'$. Since $\rho_\lambda(\nu)$ is defined only for $\nu\in I'_{D'}$, σ' must correspond to the identity element of the Weyl group, i.e. (14) takes the form

$$\rho_\lambda(\nu)=\sum_{\sigma\in W}\operatorname{sgn}\sigma P^{(\Delta'_+)}[\sigma(\beta+\lambda)-(\beta+\nu)] \tag{15}$$

for $\nu\in I\cap I_{D'}=I_{D'}$, and $\rho_\lambda(\nu)=0$ for $\nu\in I'_{D'}$.

Thus we finally obtain:

The multiplicity $\rho_\lambda(\lambda')$ of the irreducible representation $\pi'_{\lambda'}$ of the subalgebra g', contained in the irreducible representation π_λ of the algebra g, is computed by the formula

$$\rho_\lambda(\lambda')=\sum_{\sigma\in W}\operatorname{sgn}\sigma P^{(\Delta'_+)}[\sigma(\beta+\lambda)-(\beta+\lambda')] \tag{16}$$

for $\lambda'\in I_{D'}$, and $\rho_\lambda(\lambda')=0$ for $\lambda'\in I'_{D'}$; $P^{(\Delta'_+)}(\mu)$ is the function of decomposition of the vector μ into a sum of positive roots of the algebra g which do not belong to the subalgebra g'.

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Note: Figure translations are in progress. See original paper for figures.