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Abstract

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ON THE LIMIT SEQUENCE OF AN EXACT SEQUENCE OF INVERSE SPECTRA

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In this note, by methods of homological algebra, we study the question of the cases in which an exact sequence of spectral groups of homologies is exact.

Let I be an arbitrary directed set, and let inverse spectra of abelian groups $\{A^n_\alpha, (\varphi^\beta_\alpha)^n\}_{\alpha \in I,\ n=0,1,...}$, together with homomorphisms $d^n_\alpha : A^n_\alpha \to A^{n+1}_\alpha$, form an exact sequence of inverse spectra. In other words, let

$$(\varphi^\beta_\alpha)^{n+1} d^n_\beta = d^n_\alpha (\varphi^\beta_\alpha)^n \quad (\beta > \alpha,\ n = 0, 1, ...)$$

and let the sequences

$$0 \to A^0_\alpha \to A^1_\alpha \to \cdots \to A^n_\alpha \to \cdots$$

be exact for every $\alpha \in I$.

Let $\varprojlim^{(i)}$ denote the i-th right derived functor of the functor $\varprojlim$, which assigns to each inverse spectrum with directed set I its projective limit. The sequences

$$0 \to \varprojlim^{(i)} A^0_\alpha \to \varprojlim^{(i)} A^1_\alpha \to \cdots \to \varprojlim^{(i)} A^n_\alpha \to \cdots, \qquad (L_i)$$

obtained by applying the functors $\varprojlim^{(i)}$ to the exact sequence of spectra

$$0 \to \{A^0_\alpha\} \to \{A^1_\alpha\} \to \cdots \to \{A^n_\alpha\} \to \cdots, \qquad \text{(M)}$$

are, in general, not exact, but they are complexes in the sense that the composition of any two adjacent homomorphisms in each of these sequences is equal to zero. Consequently, the groups

$$H^p \varprojlim^{(i)} A^*_\alpha$$

of homologies of these complexes are defined.

Theorem 1. *There exists a spectral sequence beginning with the term*

$$E_2^{p,q} = H^p \varprojlim^{(q)} A^*_\alpha$$

and converging to zero.

This spectral sequence is obtained as the spectral sequence of the composition of the functors [3] $Z^0 \varprojlim$, where $\varprojlim$ is regarded as a functor from the category of inverse spectra of complexes to the category of complexes, and Z^0 is the functor assigning to a complex its group of zero-dimensional cycles.

Corollary 1. *The sequence L_0 is exact at the term $\varprojlim A^n_\alpha$ if the sequences L_i are exact at the terms $\varprojlim^{(i)} A^{n-i-1}_\alpha$ for $i > 0$. In particular, the sequence L_0 is exact at the term $\varprojlim A^n_\alpha$ if $\varprojlim^{(i)} A^{n-i-1}_\alpha = 0$ for $i > 0$.*

If the spectra $\{A^n_\alpha\}$ are such that $\varprojlim^{(i)} A^n_\alpha = 0$ for $i \geq 2$, then the spectral sequence yields the natural isomorphisms

$$H^n \varprojlim A^*_\alpha = H^{n-2} \varprojlim^{(1)} A^*_\alpha.$$

Roos, who studied the functors $\lim^{(i)}_{\leftarrow}$ in notes ([4–6]), established that in the case when the directed set I coincides with the set Z_+ of positive integers, $\lim^{(i)}_{\leftarrow} A_\alpha = 0$ for $i \geq 2$ for any spectrum and $\lim^{(1)}_{\leftarrow} A_\alpha = 0$ if the projections in the spectrum $\{A_\alpha\}$ are epimorphisms. Therefore, in the case $I = Z_+$, the sequence L_0 is exact at the term $\lim_{\leftarrow} A^n_\alpha$ if the projections in the spectrum $\{A^{n-2}_\alpha\}$ are epimorphisms. This last assertion coincides with a theorem of N. Tynjanski ([8]).

In the first of his notes Roos constructs, for a direct spectrum of modules $\{A_\alpha\}$ and a module B over a fixed ring Λ, a spectral sequence beginning with the term $E^{p,q}_2 = \lim^p_{\leftarrow} \operatorname{Ext}^q_\Lambda(A_\alpha, B)$ and converging to the term $E^{p,q}_\infty$ associated with the module $\operatorname{Ext}^n_\Lambda(\lim_{\rightarrow} A_\alpha, B)$. This spectral sequence, in the case considered here of spectra of abelian groups, yields the natural exact sequence

$$0 \to \overset{(0)}{\underset{\leftarrow}{\lim}} \operatorname{Hom}(A_\alpha, B) \to \operatorname{Ext}(\underset{\rightarrow}{\lim} A_\alpha, B) \to$$

$$\to \underset{\leftarrow}{\lim} \operatorname{Ext}(A_\alpha, B) \to \overset{(2)}{\underset{\leftarrow}{\lim}} \operatorname{Hom}(A_\alpha, B) \to 0$$

and the natural isomorphisms

$$\overset{(i)}{\underset{\leftarrow}{\lim}} \operatorname{Hom}(A_\alpha, B) \approx \overset{(i-2)}{\underset{\leftarrow}{\lim}} \operatorname{Ext}(A_\alpha, B) \quad \text{for } i > 2.$$

This result of Roos serves as the starting point in the proof of the following assertions.

Theorem 2. a) A compact spectrum, i.e. a spectrum in which the groups are compact and the projections are continuous, is acyclic. b) A spectrum of finite-dimensional vector spaces over a fixed field F is acyclic. c) If an exact functor Γ from the category of inverse spectra over a directed set I to the category of inverse spectra over another directed set I' takes compact spectra to compact spectra and if, moreover, the functors $\lim_{\leftarrow I'} \Gamma$ and $\lim_{\leftarrow I}$ are naturally equivalent, then the functors $\lim^{(i)}_{\leftarrow I'} \Gamma$ and $\lim^{(i)}_{\leftarrow I}$ are naturally equivalent as well. In particular, the values of the functors $\lim^{(i)}_{\leftarrow}$ do not change when passing to a cofinal part of the spectrum and when passing to a spectrum with weakened projections.

Here the spectrum $\{A_\alpha\}$ is called acyclic if

$$\overset{(i)}{\underset{\leftarrow}{\lim}} A_\alpha = 0$$

for $i \geq 1$.

Corollary 2. If in the sequence (M) the spectra $\{A^n_\alpha\}$ are compact or are spectra of finite-dimensional vector spaces over a fixed field F, then the sequence L_0 is exact.

Let us note that here, in contrast to Theorem 5.6 of the book ([7]), the continuity of the homomorphisms d^n_α is not assumed.

Let $\mathrm{Pext}(A, B)$ denote the group of those extensions of the group B with kernel A in which the subgroup A is pure. This group coincides with the maximal complete subgroup of the group $\mathrm{Ext}(A, B)$ of all extensions of the group B with kernel A ([9]). The following is important for what follows.

Theorem 3. Let $\{A_\alpha\}$ be a direct spectrum of groups with a finite number of generators and let B be an arbitrary group; then

$$\overset{(i)}{\underset{\leftarrow}{\lim}} \mathrm{Hom}(A_\alpha, B) = 0$$

for $i \geq 2$ and

$$\overset{(1)}{\underset{\leftarrow}{\lim}} \mathrm{Hom}(A_\alpha, B) \approx \mathrm{Pext}(\underset{\rightarrow}{\lim} A_\alpha, B), \qquad \overset{(i)}{\underset{\leftarrow}{\lim}} \mathrm{Ext}(A_\alpha, B) = 0$$

for $i \geq 1$.

Corollary 3. If $\{A_\alpha\}$ is a direct spectrum of groups with a finite number of generators and B is an arbitrary group, then the mapping

$$\mathrm{Ext}(\underset{\rightarrow}{\lim} A_\alpha, B) \to \underset{\leftarrow}{\lim} \mathrm{Ext}(A_\alpha, B)$$

is an epimorphism, and the kernel of this mapping is the maximal complete subgroup of the group

$$\mathrm{Ext}(\underset{\rightarrow}{\lim} A_\alpha, B).$$

Theorem 4. Let $\{A_\alpha\}$ be an inverse spectrum of groups with a finite number of generators, and let B be an arbitrary group. Then $\overset{(i)}{\underset{\leftarrow}{\lim}}(A_\alpha \otimes B) = 0$ for $i \geq 2$,

$$\overset{(1)}{\underset{\leftarrow}{\lim}}(A_\alpha \otimes B) \approx \operatorname{Ext}(\underset{\rightarrow}{\lim} \operatorname{Hom}(A_\alpha, Z), B),$$

$\overset{(i)}{\underset{\leftarrow}{\lim}}(A_\alpha * B) = 0$ for $i \geq 2$,

$$\overset{(1)}{\underset{\leftarrow}{\lim}}(A_\alpha * B) \approx \operatorname{Pext}(\underset{\rightarrow}{\lim}\, {}_t\overline{A}_\alpha, B),$$

where ${}_t\overline{A}_\alpha$ denotes the group of characters of the maximal periodic subgroup of the group A_α.

Definition. An inverse spectrum of q-dimensional homology groups with coefficients in the group G of the nerves of finite open coverings of a bicompact pair (X, Y), whose limit, by definition, is the group $H_q(X, Y; G)$ of spectral homology of the pair (X, Y), is called a **homological spectrum**.

Let A_α be a homological spectrum. Then $\overset{(i)}{\underset{\leftarrow}{\lim}}\, A_\alpha = 0$ for $i \geq 2$,

$$\overset{(1)}{\underset{\leftarrow}{\lim}}\, A_\alpha = \operatorname{Pext}(H^q(X, Y), G).$$

In particular, if all spectra of the sequence (M) are homological spectra, then

$$H^p \underset{\rightarrow}{\lim}\, A^*_\alpha \approx H^{p-2} \operatorname{Pext}(H^*(X^*Y^*), G).$$

From this remark there follows the following main

Theorem 5. For an arbitrary finite-dimensional bicompact pair (X, Y) and coefficient group G, the sequences

$$\cdots \longrightarrow H_p(Y; G) \longrightarrow H_p(X; G) \longrightarrow H_p(X, Y; G) \longrightarrow H_{p-1}(Y; G) \longrightarrow \cdots$$

$$\cdots \longrightarrow \operatorname{Pext}(H^{p+1}(X), G) \longrightarrow \operatorname{Pext}(H^{p+1}(X, Y); G) \longrightarrow$$

$$\longrightarrow \operatorname{Pext}(H^p(Y); G) \longrightarrow \operatorname{Pext}(H^p(X); G) \longrightarrow \cdots \qquad (1)$$

have identical homology groups in the corresponding places.

Similarly, for every exact sequence of groups $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$, the sequences

$$\cdots \longrightarrow H_p(X,Y;G') \longrightarrow H_p(X,Y;G) \longrightarrow H_p(X,Y;G'') \longrightarrow$$

$$\longrightarrow H_{p-1}(H,Y;G') \longrightarrow \cdots \qquad (2)$$

$$\cdots \longrightarrow \mathrm{Pext}(H^{p+1}(X,Y),G) \longrightarrow \mathrm{Pext}(H^{p+1}(X,Y),G'') \longrightarrow$$

$$\longrightarrow \mathrm{Pext}(H^{p}(X,Y),G) \longrightarrow \mathrm{Pext}(H^{p}(X,Y),G) \longrightarrow \cdots$$

have identical homology groups in the corresponding places.

The group $\mathrm{Pext}(A,B) = 0$ if the group B is compact or if A is a direct sum of cyclic groups. If B' is a pure subgroup of the group B, then the sequence $\mathrm{Pext}(A,B') \to \mathrm{Pext}(A,B) \to \mathrm{Pext}(A,B/B') \to 0$ is exact; moreover, if G' is pure in G, then the homomorphism δ in the sequences (1) and (2) is trivial. In view of this, the following is valid.

Corollary 4. If the group $H^{p+1}(X,Y)$, $(H^{p+1}(X), H^p(Y))$ of a finite-dimensional bicompact pair is isomorphic to a direct sum of cyclic groups, then the sequence (1) is exact at the term $H_p(X;G)$ $(H_p(Y;G), H_p(X,Y;G))$, and the sequence (2) is exact at the terms $H_{p+1}(X,Y;G'')$, $H_p(X,Y;G')$, and $H_p(X,Y;G)$.

If in the sequence $0 \to G' \to G \to G'' \to 0$ the subgroup G' is pure in G, then the sequence (2) is exact at the terms $H_p(X,Y;G')$ and $H_p(X,Y;G)$.

One more application of the concepts introduced here concerns the study of the homomorphism of Sitnikov homology groups into spectral ones ([1]). Borel and Moore ([2]) defined homology groups $\mathcal{H}_*(X;G)$ of a bicompact space X, coinciding in the metric case with the Sitnikov groups. There exists a natural transformation

$$\alpha : \mathcal{H}_*(X;G) \to H_*(X;G),$$

which, together with the formulas for the universal coefficients, forms the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(H^{n+1}(X),G) & \longrightarrow & \mathcal{H}_n(X;G) & \to & \mathrm{Hom}(H^n(X),G) & \to 0 \\
& \downarrow \beta & & \downarrow \alpha & & \downarrow 1 & \\
0 \to & \varprojlim \mathrm{Ext}(H^{n+1}(X_\alpha),G) & \to & H_n(X;G) & \to & \mathrm{Hom}(H^n(X),G) & \to 0.
\end{array}$$

In this diagram the rows are exact. Obviously, $\operatorname{Ker}\alpha = \operatorname{Ker}\beta$ and $\operatorname{Coker}\alpha = \operatorname{Coker}\beta$. But $\operatorname{Coker}\beta = 0$, while $\operatorname{Ker}\beta$ coincides with the maximal complete subgroup of the group $\operatorname{Ext}(H^{n+1}(X), G)$. Thus α is an epimorphism, and its kernel is a complete group.

Apparently, there is a possibility of extending Theorem 3 to the case of spectra of modules*. In this connection the following theorem is of interest:

Theorem 6. Let Λ be a Noetherian ring. There exists a sequence of functors $T^i(A, B)$ from the category of pairs of Λ-modules to the category of modules such that, for every direct spectrum $\{A_\alpha\}$ of modules with a finite number of generators and any Λ-module B, the modules $T^i(\varinjlim A_\alpha, B)$ and

$$\varprojlim{}^{(i)} \operatorname{Hom}_\Lambda(A_\alpha, B)$$

are naturally isomorphic. These isomorphisms uniquely determine the functors T^i.

Problem. Will the direct sum of an arbitrary number of copies of a given acyclic spectrum be acyclic? This problem is equivalent to the well-known problem of Whitehead, which asks whether a group A is free if $\operatorname{Ext}(A, Z) = 0$. Since the product of acyclic spectra is acyclic, this problem is also equivalent to the following question: is the mapping

$$\varprojlim_{\leftarrow\beta} \prod \{A_\alpha\}_\beta \to \varprojlim_{\leftarrow\beta} \left[\prod_\beta \{A_\alpha\}_\beta \Big/ \sum_\beta \{A_\alpha\}_\beta \right]$$

an epimorphism if each spectrum $\{A_\alpha\}_\beta$ is isomorphic to a fixed acyclic spectrum?

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* **Note added in proof.** After the article had been submitted for typesetting, the author succeeded in obtaining such an extension of Theorem 3.

Note: Figure translations are in progress. See original paper for figures.