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Let

1965

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Abstract

Full Text

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ON THE BEHAVIOR AT INFINITY OF CERTAIN CLASSES OF POLYNOMIALS

(Presented by Academician I. G. Petrovskii, February 11, 1965)

Let

$$A(x,\xi)=\sum_{\alpha} a_{\alpha}(x)\xi^{\alpha} \tag{1}$$

be a polynomial with respect to the vector $\xi=(\xi_1,\dots,\xi_n)$, with continuous complex coefficients $a_{\alpha}(x)$ depending on the parameter $x=(x_1,\dots,x_n)\in\overline{Q}$, where Q is a certain bounded domain of m-dimensional space, $\alpha=(\alpha_1,\dots,\alpha_n)$ is an integer vector, and $\xi^{\alpha}=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. We shall assume that $a_0(x)=A(x,0)\neq 0$ for all $x\in\overline{Q}$; it will be clear from what follows that this assumption does not diminish generality. In addition, in all subsequent considerations the vector ξ will be assumed real.

Certain problems of the general theory of partial differential equations are closely connected with the study of the behavior of a general polynomial of the form (1) for real $\xi\to\infty$. One such problem will be considered below (in Sec. V).

Definition 1. A polynomial $\widetilde{A}(x,\xi)$ will be called the **principal term** of the polynomial $A(x,\xi)$ if: 1) for all $x\in\overline{Q}$,

$$A(x,\xi)=\widetilde{A}(x,\xi)(1+o(1))$$

as $\xi\to\infty$; 2) the polynomial $\widetilde{A}(x,\xi)$ is the sum of a minimal number of monomials of the polynomial $A(x,\xi)$.

It can be proved that every polynomial $A(x,\xi)$ has a unique principal term. The study of the behavior at infinity of the principal term $\widetilde{A}$ is, generally speaking, simpler than the analogous study of the polynomial A itself; it is therefore natural to single out those classes of polynomials (1) for which there exists a sufficiently simple algorithm for extracting the principal terms.

I. Let us first consider the polynomial $A(x,\xi)$ (1) with real coefficients $a_{\alpha}(x)$. Let $\mathfrak{C}(x)$ be the set of integer points of the space $\{\alpha\}$ lying in the region $\alpha_1\geq 0,\dots,\alpha_n\geq 0$ and such that $a_{\alpha}(x)\neq 0$, and let $\mathfrak{N}(x)\equiv\mathfrak{N}^n(x)$ be the minimal convex polyhedron containing the set $\mathfrak{C}(x)$. According to our assumption, the origin $O\in\mathfrak{N}(x)$. Suppose, moreover, that $\mathfrak{N}(x)\equiv\mathfrak{N}$ for all $x\in Q$. (For this condition to hold it is necessary and sufficient that $a_{e^k(x)}(x)\neq 0$ for all $x\in Q$

and for all vertices $e^k(x)$ of the polyhedron $\mathfrak{N}(x)$.) The open faces of dimension k, $0 \le k \le n-1$, will be denoted by $\mathfrak{N}_i^k$, $i = 1, \dots, N_k$, where N_k is the number of such faces. The vertices of the polyhedron $\mathfrak{N}$, i.e. the faces $\mathfrak{N}_i^{(0)}$, $i = 1, \dots, N_0$, of zero dimension, will for subsequent convenience be denoted by $e^i = \mathfrak{N}_i^{(0)}$, $i = 1, \dots, N_0$. The surface of the polyhedron $\mathfrak{N}$ will be denoted by $\mathfrak{N}^{n-1}$:

$$\mathfrak{N}^{n-1} = \bigcup_{k=0}^{n-1} \bigcup_{i=1}^{N_k} \overline{\mathfrak{N}_i^k},$$

where $\overline{\mathfrak{N}_i^k}$ is the closure of the face $\mathfrak{N}_i^k$.

Let $m^i = (m_1^i, \dots, m_n^i)$ be the vector of the normal to the face $\mathfrak{N}_i^{n-1}$, outward with respect to the polyhedron $\mathfrak{N}$.

Definition 2. A face $\mathfrak{N}_i^{n-1}$ of the polyhedron $\mathfrak{N}$ will be called **principal** if among the coordinates m_s^i, $s = 1, \dots, n$, of its outward normal at least one is positive. A point $a \in \mathfrak{N}^{n-1}$ will be called **principal** if a is a limit point for at least one principal $(n-1)$-dimensional face. A face $\mathfrak{N}_i^k$, for $k = 0, \dots, n-2$, will be called **principal** if it consists of principal points of $\mathfrak{N}^{n-1}$.

Consider, for arbitrary k and i, $k = 0, \dots, n-1$; $i = 1, \dots, N_k$, the polynomial

$$A_{\mathfrak{N}_i^k}(\xi) = \sum_{a \in \mathfrak{N}_i^k} a_a(x)\xi^a.$$

Definition 3. A face $\mathfrak{N}_i^k$, $i = 1, \dots, N_k$, $k = 0, \dots, n-1$, will be called **nondegenerate** if on the real sphere $|\xi| = 1$ the polynomial $A_{\mathfrak{N}_i^k}(x, \xi)$ can vanish only at points of intersection of this sphere with the coordinate planes.

Definition 4. The polynomial $A(x, \xi)$ (1) is called **nondegenerate** if all its principal faces $\mathfrak{N}_i^k$ are nondegenerate.

The polynomials $A_{\mathfrak{N}_i^k}(x, \xi)$ are generalized homogeneous and become homogeneous after the substitution $\xi = \zeta^{m^l}$ $(\xi_1 = \zeta_1^{m_1^l}, \dots, \xi_n = \zeta_n^{m_n^l})$, where m^l is the vector of the outward normal of one of those faces $\mathfrak{N}_l^{n-1}$ for which the face $\mathfrak{N}_i^k$ is limiting. (We note that if any of the numbers $m_i^l = 0$, then the corresponding variables ξ_i should be put equal to arbitrary constant numbers.) Therefore the verification of nondegeneracy of the polynomial $A_{\mathfrak{N}_i^k}(x, \xi)$ reduces to checking the "generalized ellipticity" of the polynomials $A_{\mathfrak{N}_i^k}(x, \xi)$ for all principal faces $\mathfrak{N}_i^k$ of the polyhedron.

Definition 5. The polynomial $A(x, \xi)$ is called **complete** if the vertex $O = (0, \dots, 0)$ of its polyhedron $\mathfrak{N}$ is not principal.

If the polynomial $A(x, \xi)$ is complete, then one can show that its polyhedron $\mathfrak{N}$ is n-dimensional, its only nonprincipal vertex is the vertex O, on each coordinate axis of the space $\{a\}$ the polyhedron $\mathfrak{N}$ has one principal vertex, and all faces of the polyhedron $\mathfrak{N}$ which are not principal lie in the coordinate planes.

Theorem 1. *If the polynomial* $A(x,\xi)$ *is complete and nondegenerate, then* $\lim_{\xi\to\infty}|A(x,\xi)| = \infty$. *Moreover* $\lim_{\xi\to\infty} A(x,\xi) = (\operatorname{sign} a_{e^k}(x))\infty$, *where* e^k *is any principal vertex of the polyhedron* $\mathfrak{N}$.

Theorem 2. *If the polynomial* $A(x,\xi)$ *is complete and nondegenerate, and* $\mathfrak{N}(x) \equiv \mathfrak{N}$ *for all* $x \in Q$, *then its principal part*

$$\widetilde{A}(x,\xi) = \sum_{\alpha\in\mathfrak{N}^{n-1}_{\mathrm{pr}}} a_\alpha(x)\xi^\alpha, \tag{2}$$

where $\mathfrak{N}^{n-1}_{\mathrm{pr}}$ *is the aggregate of all principal points of the surface* $\mathfrak{N}^{n-1}$ *of the polyhedron* $\mathfrak{N}$.

Together with the concept of the principal part of the polynomial $A(x,\xi)$, it is convenient also to introduce the somewhat cruder concept of a majorizing polynomial.

Definition 6. The polynomial $\hat{A}(x,\xi)$ is called a **majorizing polynomial** for the polynomial $A(x,\xi)$, if there exist

such constants $\sigma \geqslant 0$ and $\gamma > 0$ that the inequalities

$$\gamma^{-1}\widetilde{A}(x,\xi) \leqslant A(x,\xi) + \sigma \leqslant \gamma\widetilde{A}(x,\xi)$$

are satisfied for all real ξ and $x \in Q$.

It is clear that the principal part $\overline{A}(x,\xi)$ of the polynomial $A(x,\xi)$ is at the same time one of its majorizing polynomials.

Theorem 3. If the polynomial $A(x,\xi)$ is complete and nondegenerate, and $\mathfrak{N}(x) \equiv \mathfrak{N}$ for all $x \in Q$, then a majorizing polynomial for it is the polynomial

$$\widetilde{A}(\xi) = \xi^{e_1} + \dots + \xi^{e_{N_0}}. \tag{3}$$

II. Let us now consider the general case of complex coefficients $a_\alpha(x)$. The construction of the set $\mathfrak{C}(x)$ and of the polyhedron $\mathfrak{N}(x)$ is carried out in the same way as in the preceding point. We shall assume that $\mathfrak{N}(x) \equiv \mathfrak{N}$ for all $x \in Q$. All definitions of the preceding point are also retained.

Theorem 4. If the polyhedron $\mathfrak{N}(x)$ of the polynomial $A(x,\xi)$ with complex coefficients does not depend on $x \in Q$, and the polynomial $A(x,\xi)$ is complete and nondegenerate, then

$$\lim_{\xi\to\infty} A(x,\xi) = \infty.$$

The principal part in the polynomial $A(x,\xi)$ is the polynomial (2), and as a majorizing polynomial for the polynomial $|A(x,\xi)|^2$ one may take the polynomial $(\widetilde{A}(\xi))^2$, where $\widetilde{A}$ is defined by formula (3).

III. **Definition 7** (^{1}). A polynomial $A(\xi)$ with complex coefficients is called **hypoelliptic** if

$$\lim_{\xi\to\infty}(A(\xi))^{-1} = \lim_{\xi\to\infty}\frac{|\operatorname{grad} A(\xi)|}{A(\xi)} = 0.$$

Theorem 5. In order that a complete nondegenerate polynomial $A(\xi)$ be hypoelliptic, it is necessary and sufficient that the outward normals of all principal $(n-1)$-dimensional faces of the polyhedron $\mathfrak{N}$ have only positive coordinates.

From Theorem 5 there follows a criterion for hypoellipticity of a differential operator $A(-iD)$ with complex coefficients, if the characteristic polynomial of this operator $A(\xi)$ is complete and nondegenerate.

IV. Let

$$\mathcal{P}(x,D)u \equiv \sum_{\alpha} p_\alpha(x) D^\alpha u = f(x) \tag{4}$$

be a linear differential equation with sufficiently smooth complex coefficients $p_\alpha(x)$, where $x = (x_1,\dots,x_n) \in \overline{Q}$; Q is some bounded domain; $D = (D_1,\dots,D_n)$. Suppose that the characteristic polynomial for the operator $\mathcal{P}(x,D)$ has the form

$$\mathcal{P}(x,i\xi) \equiv \operatorname{Re}\mathcal{P}(x,i\xi) + i\operatorname{Im}\mathcal{P}(x,i\xi) = A(x,\xi) + iB(x,\xi), \tag{5}$$

$$A(x,\xi) = \sum_{\alpha} a_\alpha(x)\xi^\alpha, \qquad B(x,\xi) = \sum_{\alpha} b_\alpha(x)\xi^\alpha,$$

where the polynomial $A(x,\xi)$ is a complete nondegenerate polynomial with real coefficients. Together with the polyhedron $\mathfrak{N}(x) \equiv \mathfrak{N}$ for the polynomial $A(x,\xi)$, consider the convex polyhedron $\mathfrak{M} = (\bigcup R_\alpha) \cup \mathfrak{N}$, where

R_α is the set of those points $\beta = (\beta_1,\dots,\beta_n)$ for which $0 \le \beta_i \le \alpha_i$, $i = 1,\dots,n$. It turns out that all vertices $f^i = (f_1^i,\dots,f_n^i)$ of the polyhedron $\mathfrak{M}$ have even coordinates (f_j^i are even numbers). Introduce on the set of functions $u(x) \in C_0^\infty(\bar{Q})$ the scalar product

$$(u,v) = \int_Q \sum D^{f^i/2}u \cdot D^{f^i/2}\bar{v}\,dx.$$

The Hilbert space obtained by completing $C_0^\infty(Q)$ with respect to this scalar product will be denoted by $\mathfrak{H}(Q)$.

Definition 8. A function $u(x) \in \mathfrak{H}(Q)$ will be called a **solution of the first boundary-value problem** in the domain Q for equation (4), with $f(x) \in \mathcal{L}_2(Q)$, if

$$(u, \mathcal{P}^*(x, D)v) = (f, v)$$

for every $v \in C_0^\infty(Q)$. (Here $\mathcal{P}^*(x, D)$ is the operator formally adjoint to the operator $\mathcal{P}(x, D)$.)

Theorem 6. *The problem of finding a solution of the first boundary-value problem for equation (4) is Fredholm if the polynomial $B(x, \xi)$ (5) is subordinate to the polynomial $A(x, \xi)$. Under the same assumption, for the operator $\mathcal{P}(x, D) + \lambda$, for sufficiently large λ, the first boundary-value problem is uniquely solvable.*

Here by subordination of the polynomial $B(x, \xi)$ to the polynomial $A(x, \xi)$ we mean the following. Suppose first that the coefficients $p_\alpha(x)$ are complex. Then every point α for which $p_\alpha(x) \neq 0$ in $\bar{Q}$ must be either an interior point of $\mathfrak{M}$, or such a boundary point of $\mathfrak{M}$ that lies on no principal face.

If, however, the coefficients $\{p_\alpha(x)\}$ are real, then the subordination condition is weaker: every α for which $p_\alpha(x) \neq 0$ in $\bar{Q}$: a) either lies inside $\mathfrak{M}$; b) or, lying on the boundary of $\mathfrak{M}$, lies on none of its principal faces; c) or lies on the boundary of $\mathfrak{M}$, or even outside $\mathfrak{M}$, but $\alpha - i_k$ (if $\alpha - i_k \geq 0$) for any $k = 1, \dots, n$ lies either inside $\mathfrak{M}$, or on the boundary of $\mathfrak{M}$, but not on a principal face (here i_k is the direction vector of the k-th coordinate axis, and the inequality $\beta \geq 0$ means that $\beta_i \geq 0$, $i = 1, \dots, n$).

V. The first boundary-value problem can also be considered for a "parabolic" equation constructed with the aid of the "elliptic" operator in (4), in the same way as in Section IV of paper ([2]). In this case Theorem 4 from ([2]) remains valid as well.

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Received
22 I 1965

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Note: Figure translations are in progress. See original paper for figures.

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