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Abstract

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ON THE CONVERGENCE OF POSITIVE FUNCTIONALS AND OPERATORS

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In many mathematical problems one has to establish the convergence of sequences of linear functionals or operators on certain Banach spaces E. In doing so it is often possible to reduce the full analysis to the study of the values of the functionals or operators only on certain subspaces of the space E. In a general form, such a scheme for sequences of positive functionals and operators in the space $C[0,1]$ of functions continuous on $[0,1]$ was proposed and developed by P. P. Korovkin ([4–6]). The results of P. P. Korovkin were generalized by V. I. Volkov ([7–10]), E. N. Morozov ([11]), and L. M. Zbyshnyi ([12]) to spaces of continuous functions of two variables. Subsequently Yu. A. Shashkin ([13,14]) studied in detail the corresponding questions for spaces of functions continuous on a finite-dimensional compact set.

In the present paper positive functionals and operators are studied in a general Banach space E, semiordered by a cone K (see ([1,2])). Below, K^* denotes the set of positive linear functionals.

1. We shall say that a nonzero functional $f \in K^*$ passes through a point $x_0 \in K$ if $f(x_0) = 0$. Through every boundary point of a solid cone there passes at least one such functional; through interior points of the cone no nonzero positive functionals pass. In the case of nonsolid cones the situation is more complicated.

A nonzero point $x_0 \in K$ will be called a **point of smoothness** of the cone K if there passes through it a unique, up to norm, nonzero positive functional.

Positive functionals f_0 passing through points of smoothness possess the following remarkable property: from the convergence of sequences of positive functionals f_n to f_0 on certain two-dimensional subspaces there follows their convergence on all of E.

Theorem 1. *Let a sequence of positive linear functionals f_n satisfy the conditions*

$$\lim_{n\to\infty} f_n(x_0) = 0, \qquad \lim_{n\to\infty} f_n(x_1) = \alpha_0, \tag{1}$$

where x_0 is a point of smoothness of the solid cone K, and x_1 is an interior point of the cone K. Then the sequence f_n converges weakly on the whole space E, and

$$\lim_{n\to\infty} f_n(x) = \frac{\alpha_0}{f_0(x_1)} f_0(x) \quad (x \in E), \tag{2}$$

where f_0 is the positive functional that passes through x_0.

When passing to nonsolid cones, additional conditions arise.

Theorem 2. Let a sequence of positive linear functionals f_n satisfy the conditions

$$\lim_{n\to\infty} f_n(x_0) = 0, \qquad \lim_{n\to\infty} f_n(x_1) = \alpha_0, \tag{3}$$

where x_0 is a smoothness point of the cone K, through which passes a positive functional f_0 taking a nonzero value at the point x_1. Suppose that the norms of the functionals f_n are uniformly bounded. Then the sequence f_n converges weakly on the whole space E, and equality (2) holds.

2. Let E be the space $C(\Omega)$ of functions continuous on the compact set Ω, and let K be the cone of nonnegative functions. The smoothness points of this cone are the nonnegative functions that take the value zero at only one point. Through the smoothness points pass only functionals of the form $f(x) = x(t_0)$, where t_0 is a fixed point of the set Ω.

In the cone K of vectors with nonnegative components in the spaces l_p $(1 \le p < \infty)$, the smoothness points are the vectors for which one and only one component is equal to zero. The cone K of nonnegative functions in L_p $(1 \le p < \infty)$ has no smoothness points.

3. Denote by $F(E_0; K)$ the set of functionals $f \in K^*$ whose norm is equal to 1 and which pass through smoothness points lying in the subspace $E_0 < E$. Put

$$\|x\|_* = \sup_{f \in F(E_0;K)} |f(x)| \qquad (x \in E). \tag{4}$$

If the seminorm (4) is a norm, then the subspace E_0 will be called **saturated** (by the smoothness points of the cone K). A saturated subspace E_0 will be called **uniformly saturated** if the norm (4) is equivalent to the original norm in E.

The subspace E_0 will be called **completely saturated** if $F(E_0; K)$ contains a subset $F_0(E_0; K)$ (possibly coinciding with $F(E_0; K)$) such that:

1°. The formula

$$\|x\|_{**} = \sup_{f \in F_0(E_0;K)} |f(x)| \qquad (x \in E) \tag{5}$$

defines a norm in E that is equivalent to the original norm.

2°. Every sequence $f_n \in F_0(E_0; K)$ has a limit point in the weak topology of the form $\lambda_0 f_0$, where $\lambda_0 > 0$ and $f_0 \in F(E_0; K)$.

Theorem 3. Let the linear positive operator A coincide with the identity on some saturated subspace E_0. Then $Ax = x$ for all $x \in E$.

Theorem 4. Let K be a solid cone, and let E_0 be a completely saturated subspace. Suppose that a sequence of linear positive operators A_n converges strongly on E_0 to the identity operator. Then

$$\lim_{n\to\infty} \|A_{nx} - x\| = 0 \qquad (x \in E). \tag{6}$$

When passing to non-solid cones, one must additionally assume that the norms of the operators A_n are uniformly bounded.

We give one corollary of Theorem 5.

Theorem 5. Let K be a solid cone, and let E_0 be a completely saturated subspace. Suppose that a sequence of linear operators A_n converges strongly on E_0 and is monotone, i.e. $A_{n+1}x - A_{nx} \in K$ for $x \in K$. Then the sequence A_n converges strongly on the whole space E.

The assumption of monotonicity can be replaced by the less restrictive condition

$$A_{nx} \leqslant A_{mx} + a_0 x \qquad (n < m;\ x \in K;\ a_0 > 0). \tag{7}$$

The theorems of this section are of interest because in many cases it is possible to construct finite-dimensional saturated and completely saturated subspaces.

4. In the case of the space $C(\Omega)$ with the cone of nonnegative functions, the notions of saturation and uniform saturation coincide, but not every uniformly saturated subspace is completely saturated. In the spaces l_p $(1 \leqslant p < \infty)$ with the cone of vectors with nonnegative components, there are no uniformly saturated subspaces, but saturated subspaces do exist (even three-dimensional ones).

For finite-dimensional spaces $E = R_m$, the notions of saturation, uniform saturation, and complete saturation of a subspace coincide. In this case E_0 is a saturated subspace in R_m if in $E_0 \cap K$ one can find m distinct points x_i $(i = 1, \dots, m)$, through each of which there passes a unique supporting $(m-1)$-dimensional plane Π_i to the cone K, and if the planes $\Pi_1, \dots, \Pi_m$ are in general position. It is not hard to see that in R_m there are no $(m-1)$-dimensional saturated subspaces if the boundary of the cone K is an elliptic conical surface.

At the same time, for some cones one can indicate three-dimensional saturated subspaces. For example, such subspaces exist if the cone K is a polyhedral cone. It would be interesting to obtain a complete description of the saturated subspaces in R_m for arbitrary cones K. We give one particular result.

Theorem 6. *Let K be the cone of vectors with nonnegative components in R_m. Then three vectors*

$$e_1 = \{\xi_1, \xi_2, \dots, \xi_m\}, \quad e_2 = \{\eta_1, \eta_2, \dots, \eta_m\}, \quad e_3 = \{\zeta_1, \zeta_2, \dots, \zeta_m\}$$

form a basis of a saturated three-dimensional subspace if and only if the points

$$z_1 = \{\xi_1, \eta_1, \zeta_1\}, \quad z_2 = \{\xi_2, \eta_2, \zeta_2\}, \dots, \quad z_m = \{\xi_m, \eta_m, \zeta_m\}$$

lie on distinct one-dimensional edges of some m-hedral cone (with vertex at the origin of coordinates) in three-dimensional space.

5. In this section we consider the space $C(\Omega)$ of functions continuous on the compact set Ω, with the cone K of nonnegative functions.

Suppose that there exists a continuous mapping U of the set Ω into a set Σ, which is one-to-one on some set Ω_0 dense in Ω, and such that $U(\Omega_0)$ has no common points with $U(\Omega \setminus \Omega_0)$. Then the set Σ will be called an admissible gluing of the set Ω.

Theorem 7. *In order that there exist k-dimensional uniformly saturated subspaces in $C(\Omega)$, it is necessary and sufficient that some admissible gluing Σ of the set Ω can be homeomorphically embedded in the $(k-2)$-dimensional Euclidean sphere.*

Theorem 8. *In order that there exist k-dimensional completely saturated subspaces in $C(\Omega)$, it is necessary and sufficient that the compact set Ω can be homeomorphically embedded in the $(k-2)$-dimensional Euclidean sphere.*

The second of these theorems is contained essentially in the results of Yu. A. Shashkin ([14]).

It follows from Theorems 7 and 8 that saturated subspaces may have (as was to be expected) smaller dimension than the minimal dimension of completely saturated subspaces.

Choose in three-dimensional space five points $M_1, \dots, M_5$ so that the segments joining them do not intersect pairwise. Denote by Ω the union of these segments. The set Ω cannot be homeomorphically embedded in the two-dimensional sphere. Therefore the minimal dimension of completely saturated subspaces in $C(\Omega)$ is equal to 5. If we identify in Ω the points $M_1, \dots, M_2$, then the admissible gluing obtained can obviously be ⋯

cannot be embedded in a two-dimensional sphere. Therefore there exist four-dimensional saturated subspaces.

Similarly, if Ω is a Möbius strip, then there exist four-dimensional saturated subspaces, although the minimal dimension of fully saturated subspaces is equal to 5.

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Note: Figure translations are in progress. See original paper for figures.