I. D. PESHLETSKII

1965

SovietRxiv

Abstract

Full Text

I. D. PESHLETSKII

CLASSES OF ANALYTIC FUNCTIONS ON OPEN RIEMANN SURFACES WITH UNIVALENTLY ATTAINABLE BOUNDARY

(Presented by Academician M. A. Lavrent' ev, July 21, 1964)

In this note we set forth one approach to the consideration of analogues of the known classes of analytic functions A, D, H_δ ($\delta > 0$), given on an open Riemann surface with boundary of a certain kind.

Let R be an open Riemann surface of hyperbolic type, i.e., for it there exists a Green' s function $G(z, p)$ with pole at the point $p \in R$. We introduce its ideal boundary, using Martin' s method ([5−7]). If $K(z, p_i) = G(z, p_i)/G(p_0, p_i)$, where p_0 is some fixed point, and $\{p_i\}$ is a sequence having no limit points on R, then one says that $\{p_i\}$ determines a certain ideal boundary point when $\{K(z, p_i)\}$ converges uniformly in every compact domain on R. Two sequences $\{p_i\}$ and $\{p'_i\}$ determine one and the same ideal boundary point p if $\{K(z, p_i)\}$ and $\{K(z, p'_i)\}$ converge to one and the same function $K(z, p)$.

Denote the whole set of ideal boundary points by B; then, using the function $K(z, p)$, it is possible, in the manner defined in ([5,7]), to introduce a distance between two points p and q situated on $R + B$. With the aid of this distance the harmonic measure of boundary sets is defined ([5]).

Let σ_λ be the level lines of the function $G(z, p_0)$. We construct orthogonal trajectories $l \in L$, called below Green lines. In sufficiently general spaces of dimension $n \geq 2$, by means of Green lines various questions are solved ([1,2,4]), in particular those connected with the Dirichlet problem. Usually, in this connection, only those lines are considered which issue from the point p_0 ($L_1 \subset L$), and from Brelot' s theorem it follows that the set of Green lines intersecting all σ_λ, i.e., those for which $\inf_l G(z, p_0) = 0$, has full g-measure. This measure on the set L_1 may be introduced if one uses the one-to-one correspondence between the collection of Green lines and the directions from the point p_0; it is sufficient for this to take the magnitude of the angle of the corresponding set of directions, divided by 2π. On L_1 a topology is induced by means of the topology of a circle of sufficiently small radius with center at p_0.

Among the Green lines on the given Riemann surface R there may be some which possess a neighborhood in the indicated topology that is at the same time a simply connected domain on R, and whose boundary contains no other points of the surface except those situated on two Green lines from L_1. All the remaining Green lines we call **exceptional**; this set includes, in particular, Green lines that pass through points where $\operatorname{grad} G(z, p_0) = 0$, as well as lines that begin not at the point p_0. We adjoin to it also all Green lines for which $\inf_l G(z, p_0) \neq 0$, if they have not already fallen into it.

We introduce the following definition. An open Riemann surface $\mathfrak{R}$ is called a **surface with univalently attainable boundary** if the collection of points belonging to exceptional Green lines,

determines a part of the ideal boundary of zero harmonic measure. We shall denote the class of such surfaces by $\mathfrak{R}$.

For surfaces of this class one can construct simply connected domains D, if one considers the totality of maximal strip-like simply connected domains U_i, bounded by pairs of Green lines and by a part of the boundary B. Choose a suitable neighborhood V of the point p_0 and set $D = \bigcup_i U_i \cup V$; it can be mapped conformally onto the disk $|w| < 1$.

Examples of representatives of the class $\mathfrak{R}$ may be all finite surfaces, as well as a fairly broad set of infinitely connected domains and open Riemann surfaces of hyperbolic type of infinite genus, given, for example, in the form of covering surfaces of the z-plane. One may consider a univalently attainable part of the ideal boundary for an arbitrary open Riemann surface. A simple example when $R \notin \mathfrak{R}$ is a disk from which a suitable sequence of points has been removed, situated on some countable everywhere dense set of radii.

Since the boundary of the domain D contains a certain part $S_R \subset B$ of the ideal boundary points introduced by Martin for the surface R, which after conformal mapping is represented in the form of a set of ordinary boundary points S_D in the topology for the z-plane, the question arises of the relationship between these objects.

Theorem 1. *To every open set of ordinary boundary points on S_D there corresponds an open (in the Martin topology) set of ideal boundary points on S_R.*

To each ideal boundary point from S_R there corresponds one and only one ordinary boundary point on S_D.

The proof of this assertion could be carried out using the fact that for a plane domain with smooth boundary the points defined by Martin coincide with ordinary points ([1]). The properties of minimal functions and the specific features of surfaces of the class $\mathfrak{R}$ are also used.

If now we combine together all ideal boundary points corresponding to one nonordinary boundary point on S_D, and again call this totality an ideal boundary point, then we obtain a compactification which, apparently, does not differ essentially from the former one. Now, however, in the important special case when the univalently attainable part of the ideal boundary is already given in the form of a real line, i.e. there exists an embedding of R into another open surface R_1 such that this part of the boundary becomes a Jordan curve on R_1, we may regard the ideal boundary points as coinciding with the ordinary ones.

A single-valued analytic function $f(p)$, $p \in R$, belongs to the class $A(R)$ if $\ln^+|f(p)|$ has a harmonic majorant on R. We shall retain the usual definitions ([8]) also for the other classes $(D(R)$ and $H_\delta(R))$.

Using the distinguished domain D and the known results ([8]), one can formulate a series of assertions concerning the existence, on the univalently attainable part of the boundary of a Riemann surface, of asymptotic boundary values for various classes of functions, as well as criteria for membership of a function $f(p)$ in a given class on the surface R, based on the properties of $f(p)$ in the domain D.

Let us consider the boundary of the disk—the image of the domain D. In addition to the part S_D, it contains portions consisting of Green lines, each such portion being generated by a certain Green line reaching the boundary of the disk and splitting there into two branches which go in both directions along the circumference. It is possible that a one-to-one correspondence will be established between the points of these branches if one matches the points obtained from the intersection of Green lines with each of the σ_λ (for example, if on each branch $\inf G(z, p_0) = 0$). If one solves the welding problem ([3]), using the identification of the corresponding points, then one obtains a new ⋯

a model of the surface R in the form of the unit disk $|w'| < 1$ with a system of cuts along radii; each cut begins at an interior point of the disk, different from the center, and ends on the circumference; moreover, the Green function $G(z, p_0)$ of the surface R generates a Green function for the entire disk $|w'| < 1$, i.e. the Green lines pass into radii. The class of Riemann surfaces, each of which belongs to $\mathfrak{R}$, admitting the construction of this model, will be denoted by $\mathfrak{R}_1$. In any case, it includes all surfaces from $\mathfrak{R}$ for which on any Green line the condition

$$\inf G(z, p_0) = 0$$

is satisfied.

The simplest case when $\inf G(z, p_0) \neq 0$ is that of isolated points which do not belong to the surface but are "removable" for the Green function, i.e. after adjoining these points to the surface it is transformed into another open Riemann surface, but no changes occur in the position of the Green lines. Such points are not an obstacle to the construction of this model, but now within the disk $|w'| < 1$ there will also be contained points corresponding to the ideal boundary points from B (we shall call them **marked**).

Theorem 2. *If $R \in \mathfrak{R}_1$, then the harmonic measure of any boundary set, computed with respect to the point p_0, is equal to the g-measure of the corresponding set of Green lines.*

The proof of this assertion uses the definition of harmonic measure in the Martin topology, the properties of Green lines for $R \in \mathfrak{R}_1$, and the validity of the analogous result for the domains $D_\lambda(z \in R;\ G(z, p_0) > \lambda)$.

It follows from the theorem that the harmonic measure of any set of ideal boundary points, computed with respect to p_0, coincides with the harmonic measure of the corresponding set of points on the circumference $|w'| = 1$, computed with respect to the center of the disk. This circumstance opens up new possibilities for studying classes of analytic functions on surfaces from $\mathfrak{R}_1$. For example:

Theorem 3. *If $R \in \mathfrak{R}_1$, then a single-valued analytic function $f(p)$ belonging to the class $A(R)$ has finite asymptotic values almost everywhere with respect to harmonic measure on R.*

Because this model for R makes it possible to rely on properties of classes of analytic functions expressed in a convenient form for the disk, various theorems can be proved here that give necessary and sufficient conditions for a given function $f(p)$ to belong to one class or another, as well as analogues of a number of theorems on boundary properties known for the planar case. All constructions are simplified if there are no marked boundary points or if, for example, the boundedness of the function in their neighborhoods is known. The abovementioned dependence between harmonic measure and the g-measure of Green lines also makes it possible to note a close connection between the class $H_1(R)$ and the solvability of the Dirichlet problem formulated in terms of Green lines ([4]).

I take this opportunity to express my gratitude to Prof. L. I. Volkovyskii for valuable remarks and great attention to my work.

Perm State University
named after A. M. Gorky

Received
21 VII 1964

CITED LITERATURE

1. M. Brelot, *Lectures Functions of a Complex Variable.* Ann. Arber. Univ. Michigan Press, 1955, pp. 105–121.

2. M. Brelot, G. Choquet, Ann. Inst. Fourier Grenoble, **3**, 199 (1952).

3. L. I. Volkovyskii, Tr. Matem. instituta im. V. A. Steklova AN SSSR, **34** (1950).

4. K. Endl, Proc. Intern. Coll. on the Theory of Functions, Helsinki, 1958.

5. Z. Kuramochi, P. Fac. Sci. Hokkaido Univ., S. 1, **16**, No. 1/2, 5 (1962).

6. R. S. Martin, Trans. Am. Math. Soc., **49**, 137 (1941).

7. M. Parreau, Ann. Inst. Fourier, **3**, 103 (1951).

8. G. P. Tumarkin, S. Ya. Khavinson, Classes of analytic functions in multiply connected domains, in: *Studies on Contemporary Problems of the Theory of Functions of a Complex Variable*, Moscow, 1960, pp. 45–77.

Note: Figure translations are in progress. See original paper for figures.