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Abstract

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ASYMPTOTICS OF CLASSICAL GREEN FUNCTIONS IN THE "VISCOUS HYDRODYNAMIC APPROXIMATION"

(Presented by Academician N. N. Bogolyubov on 19 IV 1965)

In the note ([1]) a closed equation was obtained for the basic Green function for a system of elastic spheres:

$$G_{E\vec{v}}(\mathbf{p}_1, \mathbf{p}')F_1^{(0)}(\mathbf{p}_1)\left\{\frac{\mathbf{p}_1\vec{v}}{m} - E\right\} = EG_{0\vec{v}}(\mathbf{p}_1, \mathbf{p}')F_1^{(0)}(\mathbf{p}_1)\mp$$

$$\mp inr_0^2 \int_{(\mathbf{p}_2)} d\mathbf{p}_2 \int_{(\mathbf{g}\cdot\mathbf{j})>0} d\mathbf{j}\left(\frac{\mathbf{p}_2 - \mathbf{p}_1}{m}\cdot\mathbf{j}\right)F_1^{(0)}(\mathbf{p}_1)F_1^{(0)}(\mathbf{p}_2)\left\{G_{E\vec{v}}(\mathbf{p}_1^*, \mathbf{p}')+\right.$$

$$\left.+G_{E\vec{v}}(\mathbf{p}_2^*, \mathbf{p}')e^{ir_0(\vec{v}\cdot\mathbf{j})} - G_{E\vec{v}}(\mathbf{p}_1, \mathbf{p}') - G_{E\vec{v}}(\mathbf{p}_2, \mathbf{p}')e^{-ir_0(\vec{v}\cdot\mathbf{j})}\right\}. \quad (1)$$

We investigate this equation first in the simplest case, when $r_0 \ll \bar{v}_{\text{av}}/E \ll \lambda$, where $\bar{v}_{\text{av}} = \sqrt{2\theta/m}$, and the mean free path is $\lambda \sim 1/nr_0^2$. Using the method of successive approximations, we directly obtain

$$G_{E\vec{v}}(\mathbf{p}_1, \mathbf{p}') = \frac{EG_{0\vec{v}}(\mathbf{p}_1, \mathbf{p}')}{\mathbf{p}_1\vec{v}/m - E(\mp)ir_0^2 nA(\mathbf{p}_1, \mathbf{p}')}; \quad (2)$$

here

$$A(\mathbf{p}_1, \mathbf{p}') = \int_{(\mathbf{p}_2)} d\mathbf{p}_2 \int_{(\mathbf{g}\cdot\mathbf{j})>0} d\mathbf{j}\left(\frac{\mathbf{p}_2 - \mathbf{p}_1}{m}\cdot\mathbf{j}\right)F_1^{(0)}(\mathbf{p}_2).$$

As an example of the use of this solution, let us determine the correlation function of the form

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$\langle n_\nu(\mathbf{p}_1, \mathbf{p}', t) \cdot n_\nu(\mathbf{p}_1, \mathbf{p}', \tau)\rangle$, $\nu \neq 0$. For this purpose we shall apply the method developed by us in ([2]). Then, in the adopted approximation, we shall have

$$\langle n_{\vec{\nu}}(\mathbf{p}_1, \mathbf{p}', t) \cdot n_{\vec{\nu}}(\mathbf{p}_1, \mathbf{p}', \tau)\rangle =$$

$$= \frac{n\delta(\mathbf{p}_1 - \mathbf{p}')}{(2\pi)^3} F_1^{(0)}(\mathbf{p}_1) \exp\left[-i\frac{\mathbf{p}_1\vec{v}}{m}(t-\tau) - |t-\tau|\, nr_0^2 A(\mathbf{p}_1, \mathbf{p}')\right] +$$

$$+ n^2 F_1^{(0)}(\mathbf{p}_1) F_1^{(0)}(\mathbf{p}').$$

Now let us pass to the case $\bar{v}_{\text{av}}/E \gg \lambda$, corresponding to the hydrodynamic approximation ([3]). For this purpose, putting

$$\varphi_{E\vec{v}}(\mathbf{p}) = F_1^{(0)}(\mathbf{p}) \int G_{E\vec{v}}(\mathbf{p}, \mathbf{p}') \mathfrak{F}(\mathbf{p}')\, d\mathbf{p}',$$

we obtain from (1)

$$-E\varphi_{E\vec{v}}(\mathbf{p}) + \sum_\alpha \frac{p_1^\alpha v^\alpha}{m} \varphi_{E\vec{v}}(\mathbf{p}) = E\varphi_{0\vec{v}}(\mathbf{p}) + L\left(F_1^{(0)} \mid \varphi_{E\vec{v}}\right), \tag{3}$$

where L is the collision operator corresponding to the integral term on the right-hand side of (1), while $\mathfrak{F}(\mathbf{p}')$ is a regular function, whose choice will be made later.

From this equation there follow directly 5 equations of the type of transport equations:

$$-E\rho^*_{\vec{E}v} + \rho_0 \sum_\alpha v^\alpha u^*_\alpha(E; \vec{v}) = Em \int \varphi_{0\vec{v}}(\mathbf{p})\, d\mathbf{p},$$

$$-E\rho_0 u^*_i(E; \vec{v}) + \sum_\alpha v^\alpha \hat{P}_{i\alpha}(E; \vec{v}) = E \int p_i \varphi_{0\vec{v}}(\mathbf{p})\, d\mathbf{p}, \tag{4}$$

$$-E\hat{\varepsilon}_{\vec{E}v} + \sum_\alpha v^\alpha \hat{q}_\alpha(E; \vec{v}) = E \int \frac{\mathbf{p}^2}{2m} \varphi_{0\vec{v}}(\mathbf{p})\, d\mathbf{p},$$

in which the following notation has been used: $\rho_0 = mN/V$,

$$\rho^*_{\vec{E}v} = m \int \varphi_{\vec{E}v}(\mathbf{p})\, d\mathbf{p}; \qquad \rho_0 u^*_\alpha(E; \vec{v}) = \int p_\alpha \varphi_{\vec{E}v}(\mathbf{p})\, d\mathbf{p};$$

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$$\hat{\varepsilon}_{\vec{E}v} = \int \frac{\mathbf{p}^2}{2m}\varphi_{\vec{E}v}(\mathbf{p})\,d\mathbf{p}; \qquad \hat{q}_\alpha(E;\vec{v}) = \int \frac{p_\alpha}{m}\frac{\mathbf{p}^2}{2m}\varphi_{\vec{E}v}(\mathbf{p})\,d\mathbf{p}; \tag{5}$$

$$\hat{P}_{i\alpha}(E;\vec{v}) = \int \frac{p_i p_\alpha}{m}\varphi_{\vec{E}v}(\mathbf{p})\,d\mathbf{p}.$$

With the aim of obtaining an analogue of the equations of hydrodynamics in Green functions, let us use the ideas of Chapman–Enskog ([4]). Put

$$\varphi_{\vec{E}v}(\mathbf{p}) = \varphi^{(0)}_{\vec{E}v}(\mathbf{p}) + \varphi^{(1)}_{\vec{E}v}(\mathbf{p})$$

and require that

$$\int \varphi^{(1)}_{\vec{E}v}(\mathbf{p})\,d\mathbf{p} = \int p_\alpha \varphi^{(1)}_{\vec{E}v}(\mathbf{p})\,d\mathbf{p} = \int \mathbf{p}^2 \varphi^{(1)}_{\vec{E}v}(\mathbf{p})\,d\mathbf{p} = 0,$$

i.e., the introduced quantities $\rho^*_{\vec{E}v}$, $u^*_\alpha(E;\vec{v})$, $\hat{\varepsilon}_{\vec{E}v}$ are determined only with the aid of $\varphi^{(0)}_{\vec{E}v}(\mathbf{p})$.

It should be emphasized that, in order to obtain the desired asymptotics of the Green functions, it will suffice for us, in accordance with the very construction of the initial equations (1), to restrict ourselves only to the "acoustic approximation" ([5]). Therefore, using the analogy with the corresponding expressions of the calculation of the hydrodynamic approximation, one can observe that the function $\varphi^{(0)}_{\vec{E}v}(\mathbf{p})$ should be taken in the form

$$\varphi^{(0)}_{\vec{E}v}(\mathbf{p}) = f_0^M(\mathbf{p})\left\{\frac{1}{\rho_0}\rho^*_{\vec{E}v} + \sum_\alpha \frac{1}{\theta}p_\alpha u^*_\alpha(E;\vec{v}) + \left(\theta^{-2}\frac{\mathbf{p}^2}{2m} - \frac{3}{2}\theta^{-1}\right)\theta^*_{\vec{E}v}\right\}.$$

$$f_0^M(\mathbf{p}) = \frac{N}{V}(2m\pi\theta)^{-3/2}e^{-\mathbf{p}^2/2m\theta}.$$

In the zeroth approximation, according to the definitions (5), carrying out the necessary integration, we obtain

$$\hat{\varepsilon}^{(0)}_{\vec{E}v} = \frac{3}{2}\frac{\theta}{m}\rho^*_{\vec{E}v} + \frac{3}{2}\frac{\rho_0}{m}\theta^*_{\vec{E}v}; \qquad \hat{q}^{(0)}_\alpha(E;\vec{v}) = \frac{5}{2}\frac{\rho_0\theta}{m}u^*_\alpha(E;\vec{v});$$

$$\hat{P}^{(0)}_{i\alpha}(E;\vec{v}) = \delta_{i\alpha}\left\{\frac{\theta}{m}\rho^*_{\vec{E}v} + \frac{\rho_0}{m}\theta^*_{\vec{E}v}\right\}.$$

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Let us note that, substituting these expressions into equation (4), we obtain equations in which the analogy with the linearized Euler equations is directly apparent; however, in contrast to the latter, they will already be inhomogeneous.

In constructing the first approximation let us put $\varphi^{(1)}_{\vec{E}v}(\mathbf{p}) = f_0^M(\mathbf{p})\psi_{\vec{E}v}(\mathbf{p})$. Then it can be shown that the equation for $\psi_{\vec{E}v}(\mathbf{p})$ is an integral equation of the Chapman—Enskog equation type ([4]).

Indeed, retaining, as usual, in the collision integral the terms linear in $\psi_{\vec{E}v}(\mathbf{p})$ (i.e. in $\varphi^{(1)}_{\vec{E}v}(\mathbf{p})$, and in the left-hand side neglect-

···neglecting the gradients of the introduced quantities $\rho^*_{E\vec{v}}$, $u^*_\alpha(E;\vec{v})$, $\theta^*_{E\vec{v}}$ (bearing in mind the hydrodynamic approximation), i.e., assuming $\varphi^{(0)}_{E\vec{v}} \gg \varphi^{(1)}_{E\vec{v}}$, we have (for example, in the case $\operatorname{Im} E > 0$)

$$i\left(-E+\sum_\alpha \frac{p_\alpha v^\alpha}{m}\right)\varphi^{(0)}_{E\vec{v}}(\mathbf{p}) = iE\varphi_{0v}(\mathbf{p}) + L(f_0^M \mid \psi_{E\vec{v}}).$$

Following the usual procedure, we express the left-hand side of this equation with the aid of the equations of the "zeroth" approximation:

$$i\left(-E+\sum_\alpha \frac{p_\alpha v^\alpha}{m}\right)\varphi^{(0)}_{E\vec{v}}(\mathbf{p}) =$$

$$= f_0^M(\mathbf{p})\left\{ i\left[\sum_\alpha \frac{p_\alpha v^\alpha}{m}\frac{1}{\theta}\left(\frac{\mathbf{p}^2}{2m\theta}-\frac{5}{2}\right)\theta^*_{E\vec{v}} + \right.\right.$$

$$\left. + \frac{1}{m\theta}\sum_{\alpha,\beta} v^\alpha u^*_\beta(E;\vec{v})\left(p_\alpha p_\beta - \frac{1}{3}\delta_{\alpha\beta}\mathbf{p}^2\right)\right] +$$

$$\left. + iE\left[\frac{X}{\rho_0}\left(\frac{5}{2}m-\frac{\mathbf{p}^2}{2\theta}\right) + \frac{Z}{\rho_0\theta}\left(\frac{\mathbf{p}^2}{3\theta}-m\right) + \sum_\alpha \frac{Y_\alpha p_\alpha}{\rho_0\theta}\right]\right\},$$

where

$$X = \int \varphi_{0v}(\mathbf{p})\,d\mathbf{p}; \qquad Y_\alpha = \int p_\alpha \varphi_{0v}(\mathbf{p})\,d\mathbf{p}; \qquad Z = \int \frac{\mathbf{p}^2}{2m}\varphi_{0v}(\mathbf{p})\,d\mathbf{p}.$$

Being interested in slow ("hydrodynamic") processes, i.e., the case of small E and $\vec{v}$, we may here neglect the term proportional to E (as will be seen below, its contribution has practically no effect on the final results). Then the equation

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for the function $\psi_{Ev}(\mathbf{p})$ reduces to the equation investigated by Chapman and Enskog ([4]).

After performing the corresponding operations we obtain

$$\hat{q}_\alpha^{(1)} = -i\chi v_\alpha \theta^*_{E\vec{v}},$$

$$\hat{P}^{(1)}_{j\alpha}(E;\vec{v}) = -2\mu\left\{\frac{1}{2}\left(iv_\alpha u_j^*(E;\vec{v}) + iv_j u_\alpha^*(E;\vec{v})\right) - \frac{1}{3}\delta_{j\alpha} i\sum_\beta v_\beta u_\beta^*(E;\vec{v})\right\}.$$

Here the coefficients of viscosity μ and thermal conductivity χ have the standard form. Substituting the expressions obtained for $\hat{q}_\alpha$, $\hat{P}_{i\alpha}$ into equation (4), we arrive at a system of "inhomogeneous" linear algebraic equations in the Navier-Stokes approximation.

The solutions of the system have the form of ratios of two determinants and may be represented in the form

$$\theta^*_{E\vec{v}} = \frac{Q(E;\vec{v})}{E^2 - c_0^2 v^2 + i\Gamma(c_0 v;\vec{v}) + \Omega}; \qquad \rho^*_{E\vec{v}} = \frac{R(E;\vec{v})}{E^2 - c_0^2 v^2 + i\Gamma(c_0 v;\vec{v}) + \Omega};$$

$$\xi = \sum_\alpha v_\alpha u_\alpha^*(E;\vec{v}) = \frac{B(E;\vec{v})}{E^2 - c_0^2 v^2 + i\Gamma(c_0 v;\vec{v}) + \Omega},$$

where the speed of sound is

$$c_0 = \left(\frac{5}{3}\frac{\theta}{m}\right)^{1/2},$$

$$\Omega(E;\vec{v}) = \frac{4}{3}\mu\chi\frac{2}{3}\frac{m}{\rho_0^2}v^4,$$

$$\Gamma(E;\vec{v}) \simeq \Gamma(c_0 v;\vec{v}) = \left(\frac{2}{3}\chi\frac{m}{\rho_0} + \frac{4}{3}\mu\frac{1}{\rho_0}\right)c_0 v^3 - \chi\frac{2}{5}\frac{\theta}{\rho_0}v^3$$

(we have taken into account that the damping $\Gamma(E;\vec{v})$ is substantial near the resonance $E = c_0 v$).

The solutions obtained above directly determine the asymptotic ([5]) ($E \ll 1/T$; $v \ll 1/\lambda$, T is the relaxation time) expressions for a whole set of Green' s functions directly connected with the physical characteristics of the system.

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Indeed, using the definitions introduced by us, and assuming, for example, $\mathcal{F}(p') = 1, p'_\alpha$, respectively, we have

$$\langle\langle j_\alpha(r);\, n(r')\rangle\rangle_E = \rho_0 \int d\vec{v}\, e^{i\vec{v}(r-r')}\, u^*_\alpha(E;\vec{v}\mid \mathcal{F}(p')=1),$$

$$\langle\langle n(r);\, j_\alpha(r')\rangle\rangle_E = -\frac{1}{m}\int d\vec{v}\, e^{i\vec{v}(r-r')}\, \rho^*_{E\vec{v}}(\mathcal{F}(p')=p'_\alpha),$$

$$\langle\langle e(r);\, j_\alpha(r')\rangle\rangle_E = \frac{3}{2}\theta\,\langle\langle n(r);\, j_\alpha(r')\rangle\rangle_E - \frac{3}{2}\frac{\rho_0}{m}\int d\vec{v}\, e^{i\vec{v}(r-r')}\, \theta^*_{E\vec{v}}(\mathcal{F}(p')=p'_\alpha),$$

where $e(r) = \sum_i \frac{\mathbf{p}^2}{2m}\delta(q_i - r)$.

Expressions for the corresponding correlation functions can be obtained directly in accordance with the general procedure developed earlier ([2]).

We also see that the zeros of the denominators of these Green functions determine the dispersion of acoustic excitations in the system, while the damping coefficients are expressed in a natural way through the kinetic coefficients of viscosity μ and thermal conductivity $\varkappa$.

The generalization of the results obtained to the case of quantum systems presents no difficulties.

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