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Abstract

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ON DEFORMATIONS OF FIBERED SPACES

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In this paper a complete holomorphic family of deformations of a holomorphic principal fibered space with compact base is constructed. The main role in the construction is played by the differential forms considered in ([1]); it is analogous to the construction of a complete family of deformations of a compact complex manifold due to Kuranishi ([2]) (see also Nirenberg' s survey on applications of differential equations to geometry).

Let X be a complex manifold, G a complex Lie group, and M some connected complex space. By a **holomorphic family of holomorphic principal fibered spaces with base X, group G, and parameter space** M we shall mean an arbitrary holomorphic principal bundle $\mathcal{P}$ with base $X \times M$ and group G. For $t \in M$ denote by P_t the bundle over X induced by restricting the bundle $\mathcal{P}$ to $X \times \{t\}$ under the natural identification $X \leftrightarrow X \times \{t\}$. We shall write $\mathcal{P} = \{P_t\}_{t \in M}$. If $o \in M$, one also says that $\mathcal{P}$ is a holomorphic family of deformations of the bundle P_o.

The family $\mathcal{P}$ is called **complete** at the point $o \in M$ if, for every holomorphic family of deformations $\mathcal{Q} = \{Q_s\}_{s \in N}$ of the bundle $P_o = Q_{o'}$, parametrized by some complex space $N \ni o'$, there exists a holomorphic mapping $\varphi : U \to M$, defined in a neighborhood $U \subset N$ of the point o', such that $\mathcal{P}$ induces the bundle $\mathcal{Q}$ over $X \times U$ under the mapping $X \times U \to X \times M$ that takes (x, s) to $(x, \varphi(s))$ *.

Let P be some holomorphic principal fibered space with base X and group G. Then every bundle over X obtained from P as a result of a deformation will be differentiably equivalent to the bundle P. Therefore, for constructing families of deformations of the bundle P, it is useful to consider the set of all holomorphic principal fibered spaces over X that are differentiably equivalent to the bundle P. As shown in ([1]), this set can be described in the following way in terms of differential forms.

Denote by $\mathfrak{g}$ the Lie algebra of the group G, and by $\operatorname{Ad} P$ the vector fibered space with base X and fiber $\mathfrak{g}$, generated by the adjoint representation of the group G. Let $L^{p,q}(P)$ be the space of forms of type (p, q) with values in $\operatorname{Ad} P$. Denote by $Z(P)$ the set of forms $a \in L^{0,1}(P)$ satisfying the equation

$$d''a + \frac{1}{2}[a, a] = 0 \tag{1}$$

(the commutation of forms is meaningful, since $\operatorname{Ad} P$ is a bundle of Lie algebras). Further, let $\operatorname{Int} P$ be the bundle with base X and fiber G, defined by the action of the group G on itself by means of inner

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* In the paper of Kodaira and Spencer ([3]), families are considered for which the parameter space M is a manifold. However, a complete family of deformations of this kind may fail to exist. Kodaira and Spencer also consider families of bundles over a family of complex manifolds. For simplicity we restrict ourselves here to the case in which the base X is fixed.

automorphisms. Let $D(P)$ be the group of all differentiable sections of the bundle $\operatorname{Int} P$. Then $D(P)$ acts on $L^{0,1}(P)$ by means of the following operators $C(s)$ $(s \in D(P))$:

$$C(s)\alpha = \operatorname{Ad}(s^{-1})\alpha + s^{-1}d''s \qquad (\alpha \in L^{0,1}(P)).$$

Denote by $\mathcal{O}_G$ the sheaf of germs of holomorphic mappings $X \to G$. The cohomology set $H^1(X, \mathcal{O}_G)$ is identified with the set of all holomorphic principal bundles with base X and group G (considered up to equivalence). It turns out that there exists a mapping

$$p: \quad Z(P) \to H^1(X, \mathcal{O}_G),$$

whose image consists of all bundles differentiably equivalent to the bundle P, and the full inverse images of points of the image are the orbits of the group $D(P)$. We have $p(0) = P$.

Introduce in the space $L(P) = \sum_{p,q \geq 0} L^{p,q}(P)$ the topology based on uniform convergence on every compact set of the coefficients of forms and of all their partial derivatives. Then $L(P)$ becomes a Fréchet space. The following lemma makes it possible to construct holomorphic families of deformations of the bundle P.

Lemma 1. Let M be a connected complex space and $\varphi : M \to L^{0,1}(P)$ a holomorphic mapping, with $\varphi(M) \subset Z(P)$ and $0 \in \varphi(M)$. Then there exists, and moreover uniquely, a holomorphic family $\{P_t\}_{t \in M}$ of deformations of the bundle P such that $p(\varphi(t)) = P_t$ $(t \in M)$. Conversely, if $\{P_t\}_{t \in M}$ is a holomorphic family of deformations of the bundle $P = P_o$, then there exist a neighborhood U of the point o in M and a holomorphic mapping $\varphi : U \to L^{0,1}(P)$ such that $\varphi(U) \subset Z(P)$ and $p(\varphi(t)) = P_t$ $(t \in U)$.

We now proceed to the construction of a complete holomorphic family of deformations of the holomorphic principal bundle space P, under the assumption that the base X is compact. In the usual way we introduce a Hermitian scalar product in the space $L(P)$ and consider the operator δ, adjoint to d'', and the operator $\Box = d''\delta + \delta d''$ (4). Denote by G the Green operator for $\Box$, and by H the operator of orthogonal projection onto the space of harmonic forms, i.e. forms α for which $\Box\alpha = 0$.

First consider the nonlinear equation

$$\alpha = \alpha_0 - \frac{1}{2}\,\delta G[\alpha, \alpha], \tag{2}$$

where α_0 is a fixed form from $HL^{0,1}(P)$. By the method of successive approximations one verifies that, for sufficiently small α_0, equation (2) has a unique solution $\alpha(\alpha_0) \in L^{0,1}(P)$, close to zero, and $\alpha(\alpha_0)$ depends holomorphically on α_0. It can be checked that the solution α of equation (2) satisfies (1) if and only if $H[\alpha, \alpha] = 0$.

If $\alpha = \alpha(\alpha_0)$ is the holomorphic function considered above, defined in a neighborhood of zero $V \subset HL^{0,1}(P)$, then the equation

$$H[\alpha(\alpha_0), \alpha(\alpha_0)] = 0$$

defines in V a certain analytic set M. The correspondence $\alpha_0 \mapsto \alpha(\alpha_0)$ gives us a holomorphic mapping $\varphi : M \to L^{0,1}(P)$, with $\varphi(M) \subset Z(P)$ and $\varphi(0) = 0$. According to Lemma 1, φ generates a holomorphic family $\mathfrak{P} = \{P_t\}_{t\in M}$ of holomorphic principal bundles, containing $P_o = P$.

Theorem 1. The family $\mathfrak{P}$ is complete at the point $0 \in M$.

The proof rests essentially on the following lemma, which follows from the infinite-dimensional theorem on implicit functions. Denote by S the subspace of forms $\alpha \in L^{0,1}(P)$ for which $\delta\alpha = 0$.

Lemma 2. There exist neighborhoods of the zeros $W' \subset L^{0,0}(P)$ and $W'' \subset S$ such that forms of the form $C(\exp f)\alpha$ ($f \in W'$, $\alpha \in W''$) fill a neighborhood of zero $W \subset L^{0,1}(P)$. If, in addition, one restricts oneself to sections $f \in L^{0,0}(P)$ orthogonal to the holomorphic sections, then the representation $\omega = C(\exp f)\alpha$ ($\omega \in W$) is unique, and f and α depend continuously on ω.

Let now $\mathfrak{Q} = \{Q_s\}_{s\in N}$ be an arbitrary holomorphic family of holomorphic principal bundles, with $Q_o = P$. According to Lemma 1, there exist a neighborhood U of the point o in N and a holomorphic mapping $\psi : U \to L^{0,1}(P)$ such that $\psi(U) \subset Z(P)$ and $p(\psi(s)) = Q_s$ ($s \in U$). From Lemma 2 it is seen that there exist (possibly in a smaller neighborhood U) holomorphic mappings $s \to f(s) : U \to L^{0,0}(P)$ and $s \to \alpha(s) : U \to S$, where

$$C(\exp f(s)) \cdot \alpha(s) = \psi(s).$$

Then we have $\delta\alpha(s) = 0$ and $d''\alpha(s) + \frac{1}{2}[\alpha(s), \alpha(s)] = 0$ $(s \in U)$. Since for any form a the equality

$$a = Ha + d''\delta Ga + \delta d'' Ga$$

holds, the forms $\alpha(s)$ satisfy equation (2) with $a_0 = H\alpha(s)$. Consequently, $H\alpha(s) \in M$. Thus we have obtained a holomorphic mapping $s \to H\alpha(s)$ of the space U into M. It is easy to verify that $\mathfrak{Q}$ is induced by the family $\mathfrak{P}$ under this mapping. Thus Theorem 1 is proved.

Let us consider some special cases.

A. Let $HL^{0,2}(P) = 0$. Then M is a neighborhood of zero in $HL^{0,1}(P)$, and $\mathfrak{P}$ is a holomorphic family in the sense of Kodaira-Spencer.

B. Suppose that for every form $\alpha \in HL^{0,1}(P)$ we have $[\alpha, \alpha] \in HL^{0,2}(P)$. In this case it turns out that the mapping $\varphi : M \to L^{0,1}(P)$ constructed above is the identity, and M is singled out in a neighborhood $V \subset HL^{0,1}(P)$ by the equation $[\alpha, \alpha] = 0$. To prove this assertion, note that the iterative process by which equation (2) can be solved has the following form:

$$\alpha^{(r+1)} = \alpha_0 - \frac{1}{2}\delta G[\alpha^{(r)}, \alpha^{(r)}], \qquad \alpha^{(1)} = \alpha_0.$$

Since $\alpha^{(1)}$ is harmonic, $G[\alpha^{(1)}, \alpha^{(1)}] = 0$, whence $\alpha^{(2)} = \alpha_0$. Similarly $\alpha^{(3)} = \alpha_0$, etc.

C. Let P be the direct product $X \times G$. Then $L^{p,q} = L^{p,q}(P)$ is the space of forms of type (p, q) on X with values in the algebra $\mathfrak{g}$. Suppose that X is a compact Kähler manifold. Then $HL^{0,q}$ is the space of antiholomorphic forms of degree q, i.e., forms locally representable in the form

$$\alpha = a_{i_1 \cdots i_q} d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_q},$$

where $a_{i_1 \cdots i_q}$ are antiholomorphic functions with values in $\mathfrak{g}$. It is clear that the commutator of two antiholomorphic forms is again an antiholomorphic form. Therefore we are in the conditions of case B.

Thus, the complete family of deformations of the direct product $X \times G$, where X is a compact Kähler manifold, is parametrized by forms $\alpha \in HL^{0,1}$ satisfying the condition $[\alpha, \alpha] = 0$. Hence the following assertion follows.

Theorem 2. Let X be a compact Kähler manifold, G a complex Lie group. Then every holomorphic bundle obtained from the direct product $X \times G$ by means of a sufficiently small holomorphic deformation admits an integrable holomorphic connection.

Let us also note that the construction given above of the complete family of deformations carries over to the case of locally constant fiber spaces over a smooth compact manifold.

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Note: Figure translations are in progress. See original paper for figures.