ESTIMATES OF THE SPHERICAL MATRIX NORM AND OF THE CORRESPONDING LOGARITHMIC NORM

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Abstract

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S. M. LOZINSKII

ESTIMATES OF THE SPHERICAL MATRIX NORM AND OF THE CORRESPONDING LOGARITHMIC NORM

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1. **Notation.** Up to and including Sec. 5, all notation and definitions from (1) are adopted. The symbols Φ_I, Φ_{II}, Φ_{III} denote the matrix norms induced, respectively, by the 1st, 2nd, and 3rd vector norms ((2), p. 121), i.e., the norms obtained from (1), Sec. 6, for $p = +\infty$, $p = 1$, $p = 2$. Φ_{III} is often called the spherical matrix norm. The symbol γ_{III} denotes the logarithmic matrix norm corresponding to Φ_{III}. If $A \in \mathbf{M}$, then A^* denotes the matrix Hermitian-conjugate to A. If A is a square matrix, then

$$\widetilde{A} \overset{\mathrm{def}}{=} \frac{1}{2}(A + A^*).$$

If Σ denotes a simple or double sum, then Σ' denotes the sum obtained from Σ by replacing by zeros the terms with equal indices. For any $A \in \mathbf{M}$ we have (see (2), pp. 125–127)

$$\Phi_I(A) = \max_{\mu} \sum_{\nu} |a_{\mu\nu}|; \tag{1}$$

$$\Phi_{\mathrm{II}}(A) = \max_{\nu} \sum_{\mu} |a_{\mu\nu}|; \tag{2}$$

$$\Phi_{\mathrm{III}}(A) = \sqrt{\rho(AA^*)} = \sqrt{\rho(A^*A)}, \tag{3}$$

and for a square matrix A, in addition (see (3), pp. 59–60),

$$\gamma_{\mathrm{III}}(A) = \sigma(\widetilde{A}). \tag{4}$$

2. In (1), estimates were obtained for $\Phi_{\mathrm{III}}(A)$ and $\gamma_{\mathrm{III}}(A)$ in terms of the elements of the matrix A, using a decomposition of the matrix into blocks. Here we give estimates that do not use such a decomposition.

3. Let Φ be a real-valued function defined on the set of Hermitian matrices having only nonnegative eigenvalues, and such that on this set $\rho(A) \leqslant \Phi(A)$. Define on $\mathbf{M}$ the functions L_Φ and R_Φ by the formulas

$$L_\Phi(A) \overset{\text{def}}{=} \sqrt{\Phi(AA^*)}; \tag{5_1}$$

$$R_\Phi(A) \overset{\text{def}}{=} \sqrt{\Phi(A^*A)}. \tag{5_2}$$

4. **Theorem 1.** *Under the assumptions of Sec. 3, for $A \in \mathbf{M}$ we have*

$$\Phi_{\text{III}}(A) \leqslant L_\Phi(A); \tag{6_1}$$

$$\Phi_{\text{III}}(A) \leqslant R_\Phi(A). \tag{6_2}$$

Proof. $\rho(AA^*) \leqslant \Phi(AA^*)$. Applying (3), we obtain (6_1). Similarly for (6_2).

If Φ is expressed explicitly in terms of the elements of the matrix, then inequalities (6) give an estimate of $\Phi_{\text{III}}(A)$ expressed explicitly in terms of the elements of the matrix A.

For example, putting in (6) $\Phi = \Phi_I$, we obtain

$$\Phi_{\text{III}}(A) \leq \min\left(\sqrt{\Phi_I(AA^*)}, \sqrt{\Phi_I(A^*A)}\right) \quad \text{for } A \in \mathbf{M}. \tag{7}$$

Using the multiplicativity of Φ_I $((^3), p.56)$, we obtain

$$\Phi_I(AA^*) \leq \Phi_I(A)\Phi_I(A^*) = \Phi_I(A)\Phi_{II}(A),$$

which together with (7) gives

$$\Phi_{\text{III}}(A) \leq \sqrt{\Phi_I(A)\Phi_{II}(A)} \quad \text{for } A \in \mathbf{M}. \tag{8}$$

5. Define on $\mathbf{M}$ the function N by the formula

$$N(A) \overset{\text{def}}{=} \left(\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} |a_{\mu\nu}|^2\right)^{1/2} \quad \text{for } A \in \mathbf{M}_{m\times n}; \tag{9}$$

N, as is known, is a matrix norm, and $\Phi_{\text{III}} \leq N$ on $\mathbf{M}$.

6. From this point on (with the exception of item 13), we consider only square matrices of fixed order n, and the term matrix norm is used in the sense of a norm on $\mathbf{M}_n$. The remaining symbols and terms are as above.

7. If A is a square matrix, then the symbol D_A (S_A) denotes the matrix obtained from A by replacing the off-diagonal (diagonal) elements by zeros.

8. Let Φ be a nonnegative numerical function on $\mathbf{M}_n$. Define on $\mathbf{M}_n$ the function $\overline{\Phi}$ by the formula

$$\overline{\Phi}(A) \overset{\text{def}}{=} \max\left(|a_{11}|, \dots, |a_{nn}|\right) + \Phi(S_A). \tag{10}$$

9. **Theorem 2.** Let Φ and $\overline{\Phi}$ be as in item 8. Then:

1) If Φ is a matrix norm, then $\overline{\Phi}$ is a matrix norm, $\overline{\Phi}(E_n) = 1$, and $\gamma_{\overline{\Phi}}(A) = \max_k \operatorname{Re} a_{kk} + \Phi(S_A)$.
2) If $\Phi_{\text{III}} \le \Phi$ on $\mathbf{M}_n$, then $\Phi_{\text{III}} \le \overline{\Phi}$ on $\mathbf{M}_n$.
3) If Φ is a matrix norm and $\Phi_{\text{III}} \le \Phi$ on $\mathbf{M}_n$, then $\gamma_{\text{III}} \le \gamma_{\overline{\Phi}}$ on $\mathbf{M}_n$.

Proof. 1) This is proved by straightforward computations.

2) Let $\Phi_{\text{III}} \le \Phi$ on $\mathbf{M}_n$. Fix $A \in \mathbf{M}_n$. We have $A = D_A + S_A$, which gives

$$\Phi_{\text{III}}(A) \le \Phi_{\text{III}}(D_A) + \Phi_{\text{III}}(S_A) \le$$

$$\le \max_k |a_{kk}| + \Phi(S_A) = \overline{\Phi}(A).$$

3) This follows from 2) and the definition of the logarithmic norm.

Remark. The function

$$\overline{N}(A) = \max\left(|a_{11}|, \dots, |a_{nn}|\right) + N(S_A) \tag{11}$$

will be called the norm of E. Schmidt (see (4), p. 135). By Theorem 2(1) and (3),

$$\gamma_{\text{III}}(A) \le \gamma_{\overline{N}}(A) = \max_k \operatorname{Re} a_{kk} + N(S_A). \tag{12}$$

10. Let m be a natural number, $2 \le m \le n$, and let $\varkappa$ denote a fixed set of pairwise distinct integers $k_1, \dots, k_m$, where $1 \le k_1, \dots, k_m \le n$. Put

$$T_\varkappa(A) \overset{\text{def}}{=} \sum_{\mu=1}^{m} \operatorname{Re} a_{k_\mu k_\mu}; \tag{13}$$

$$N_\varkappa(A) \overset{\text{def}}{=} \left(\sum_{\mu=1}^{m} |a_{k_\mu k_\mu}|^2 + \sum_{\mu,\nu=1}^{n}{}' |a_{\mu\nu}|^2 \right)^{1/2}; \tag{14}$$

$$\Gamma_\varkappa(A) \overset{\text{def}}{=} \left([N_\varkappa(A)]^2 - \frac{[T_\varkappa(A)]^2}{m} \right)^{1/2}; \tag{15}$$

$$\alpha_\varkappa(A) \overset{\text{def}}{=} \frac{T_\varkappa(A)}{m} - \frac{1}{\sqrt{m(m-1)}}\Gamma_\varkappa(A); \tag{16}$$

$$Z_\varkappa(A) = \frac{T_\varkappa(A)}{m} + \sqrt{\frac{m-1}{m}}\,\Gamma_\varkappa(A). \tag{17}$$

If $m = n$, then the index $\varkappa$ on $T_\varkappa, N_\varkappa, \Gamma_\varkappa, Z_\varkappa$ will sometimes be omitted. Thus, for example, $T(A) = \operatorname{Re}\operatorname{tr} A$.

11. **Theorem 3.** Let $A \in M_n$. Then:

1a) If $m < n$ and

$$\alpha_\varkappa(A) \geq |a_{jj}|, \qquad \text{for } j \notin \varkappa, \tag{18}$$

then $\Phi_{\mathrm{III}}(A) \leq Z_\varkappa(A)$.

1b) If $m = n$ and $T(A) \geq N(A)$, then $\Phi_{\mathrm{III}}(A) \leq Z(A)$.

2a) If $m < n$ and

$$\alpha_\varkappa(A) \geq \operatorname{Re} a_{jj} \qquad \text{for } j \notin \varkappa, \tag{19}$$

then $\gamma_{\mathrm{III}}(A) \leq Z_\varkappa(A)$.

2b) $\gamma_{\mathrm{III}}(A) \leq Z(A)$.

3) If $m < n$ and $\beta \geq |a_{jj}|$ for $j \notin \varkappa$, then

$$\Phi_{\mathrm{III}}(A) \leq \begin{cases} \beta + \left([N_\varkappa(A)]^2 - 2\beta T_\varkappa(A) + m\beta^2\right)^{1/2}, \\ \beta + \Gamma_\varkappa(A) \quad \text{provided } \beta \geq |T_\varkappa(A)|/m. \end{cases} \tag{20}$$

4) If $m < n$ and $\beta \geq \operatorname{Re} a_{jj}$ for $j \notin \varkappa$, then the inequality is valid which is obtained from (20) by replacing Φ on the left by γ; moreover, in the second line of formula (20) one may replace $|T_\varkappa(A)|$ by $T_\varkappa(A)$.

Proof. Let the matrix D be obtained from D_A by replacing, for $j \in \varkappa$, the element a_{jj} by $\alpha_\varkappa(A)$. Then, as is easy to calculate,

$$\alpha_\varkappa(A) + N(A - D) = Z_\varkappa(A).$$

Therefore, if $m < n$ and (18) holds, then

$$\Phi_{\mathrm{III}}(A) \leq \Phi_{\mathrm{III}}(D) + \Phi_{\mathrm{III}}(A - D) \leq \alpha_\varkappa(A) + N(A - D) = Z_\varkappa(A), \tag{21}$$

which proves 1a). If $m = n$, then $T(A) \geq N(A)$ is equivalent to $\alpha_{\varkappa}(A) \geq 0$ and, consequently, the conditions of 1b) give (21); this proves 1b). If $m < n$ and (19) holds, then

$$\gamma_{\mathrm{III}}(A) \leq \gamma_{\mathrm{III}}(D) + \gamma_{\mathrm{III}}(A - D) \leq \alpha_{\varkappa}(A) + N(A - D) = Z_{\varkappa}(A), \tag{22}$$

which gives 2a). If $m = n$, then $\gamma_{\mathrm{III}}(D) = \alpha_{\varkappa}(A)$ and, consequently, (22) holds, which proves 2b).

Let $m < n$, let β be a real number, and let the matrices D_β and $\hat{D}$ be obtained from D_A by replacing, for $j \in \varkappa$, the elements a_{jj} respectively by β and $T_{\varkappa}(A)/m$. Then, if $\beta \geq |a_{jj}|$ for $j \notin \varkappa$, then $\Phi_{\mathrm{III}}(D_\beta) \leq \beta$, and $\Phi_{\mathrm{III}}(A) \leq \Phi_{\mathrm{III}}(D_\beta) + \Phi_{\mathrm{III}}(A - D_\beta) \leq \beta + N(A - D_\beta)$, whence follows the upper of inequalities (20). If also $\beta \geq |T_{\varkappa}(A)|/m$, then $\Phi_{\mathrm{III}}(\hat{D}) \leq \beta$ and

$$\Phi_{\mathrm{III}}(A) \leq \Phi_{\mathrm{III}}(\hat{D}) + \Phi_{\mathrm{III}}(A - \hat{D}) \leq \beta + N(A - \hat{D}) = \beta + \Gamma_{\varkappa}(A).$$

Assertion 3) is proved. 4) is proved analogously.

12. Obviously, $N_{\varkappa}(\tilde{A}) \leq N_{\varkappa}(A)$, and the same is true for $Z_{\varkappa}, T_{\varkappa}$; moreover, $T_{\varkappa}(A) = T_{\varkappa}(\tilde{A})$ and $\alpha_{\varkappa}(A) \leq \alpha_{\varkappa}(\tilde{A})$. Therefore, when estimating $\gamma_{\mathrm{III}}(A)$ by Theorem 3, it is recommended first to estimate $\gamma_{\mathrm{III}}(\tilde{A})$ by this theorem, and then to use the equality $\gamma_{\mathrm{III}}(A) = \gamma_{\mathrm{III}}(\tilde{A})$.
13. **Theorem 4.** $\Phi_{\mathrm{III}}(A) \leq \sqrt{Z(AA^*)}$ for $A \in M$.

Proof. $\rho(AA^*) = \sigma(AA^*) = \Upsilon_{\mathrm{III}}(AA^*)$. But, by Theorem 3 (26), $\Upsilon_{\mathrm{III}}(AA^*) \geq Z(AA^*)$.

14. Put

$$P_k \overset{\mathrm{def}}{=} \sum_{\nu}{}' |a_{k\nu}|, \qquad Q_k \overset{\mathrm{def}}{=} \sum_{\mu}{}' |a_{\mu k}|.$$

Theorem 5. If $0 \leq \alpha \leq 1$, then

$$\sigma(A) \leq \max_k (\operatorname{Re} a_{kk} + P_k^\alpha Q_k^{1-\alpha}). \tag{23}$$

Proof. The following inequality of A. Ostrovskii is known (see (5), p. 151):

$$\rho(A) \leq \max_k (|a_{kk}| + P_k^\alpha Q_k^{1-\alpha}). \tag{24}$$

Replacing A in (24) by $E_n + hA$ and applying (1), Theorem 1, we obtain (23).

15. **Example** ((5), p. 148). The eigenvalues of the matrix

$$A = \begin{bmatrix} 7+3i & -4-6i & -4 \\ -1-6i & 7 & -2-6i \\ 2 & 4-6i & 13-3i \end{bmatrix}$$

are 9, $9+9i$, $9-9i$; $\rho(A) = 9\sqrt{2} \simeq 12.7$ (+), $\sigma(A) = 9$. In (5) several estimates for $\rho(A)$ and $\sigma(A)$ were obtained, the best of which are $\rho(A) < 22.05$ and $\sigma(A) < 22.05$. Theorem 4 and Theorem 3 (2a)) give, respectively, $\Phi_{\mathrm{III}}(A) \leq \sqrt[4]{Z(AA^*)} < 18.02$ and $\Upsilon_{\mathrm{III}}(A) \leq Z(A) < 14.2$. Consequently, $\rho(A) < 18.02$, $\sigma(A) < 14.2$. Direct computation gives $\Phi_{\mathrm{III}}(A) > 17.24$, $\Upsilon_{\mathrm{III}}(A) = 13.5$.

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