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Abstract

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ON THE CALCULATION OF OSCILLATORY REGIMES
OF SOME NON-HAMILTONIAN SYSTEMS

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§ 1. **Statement of the problem.** Consider a nonlinear system (generally speaking, non-Hamiltonian) of the form

$$\dot{y} = G(y, p), \qquad \dot{p} = F(y, p). \tag{1}$$

Let the solutions of system (1) be purely periodic, and let the coordinate y have, over one period, a maximum F_1 and a minimum F_2 (F_1 and F_2 are constants). For simplicity we shall assume that over one period there is one maximum and one minimum; the case of many extrema is considered analogously. We take account of the presence of perturbing factors by introducing parameters $x = \{x_1, \ldots, x_n\}$, varying according to an arbitrary law $\dot{x} = \varepsilon X$, where $\varepsilon > 0$ is a small parameter. In addition, we introduce into the right-hand sides of (1) small nonlinear additions $\varepsilon g, \varepsilon f$. We arrive at the system

$$\begin{aligned} \dot{y} &= G(x, y, p) + \varepsilon g(x, y, p, \varepsilon), \\ \dot{p} &= F(x, y, p) + \varepsilon f(x, y, p, \varepsilon), \\ \dot{x} &= \varepsilon X(x, y, p, \varepsilon). \end{aligned} \tag{2}$$

Because of the presence of perturbing factors, the "amplitudes" F_1 and F_2 will now vary with time. We pose the following problem: to find approximations in ε for the amplitude curves F_1 and F_2, as well as approximations for the parameters x. Similar problems for a system of the form

$$\frac{d}{dt}\,[m(x)\dot{y}] + Q(x, y) = \varepsilon f(x, y, \dot{y}, \varepsilon), \tag{3}$$

$$\dot{x} = \varepsilon X(x, y, \dot{y}, \varepsilon),$$

which are a special case of system (2), under the assumption of Hamiltonian character of the degenerate system (the system obtained from (3) for $\varepsilon = 0$), were considered in ([4–7]). Some particular forms of systems of type (3) were considered in ([1–3]).

§ 2. **Main results.** To derive the equations determining the variation of the amplitude curves F_1 and F_2 and of the parameters x, we use asymptotic methods based on averaging (see ([6],[7])). Suppose that an integral of the degenerate system (1) is known:

$$p = P(x, y, C). \tag{4}$$

The amplitudes F_1 and F_2 are introduced by means of the relations:

$$G(x, F_1, P(x, F_1, C)) = 0,$$

$$G(x, F_2, P(x, F_2, C)) = 0. \tag{5}$$

Equations (5) make it possible to find the amplitude curve F_2, if the amplitude curve F_1 is known. To determine the amplitude curve F_1 and the parameters x in the first approximation, we obtain the system of equations

$$\dot{F}_1 = \frac{\varepsilon}{T} \sum_{i=1,2} \int_{F_2}^{F_1} \frac{B_i(x, F_1, y)}{G(x, y, P_i)}\, dy, \qquad \dot{x} = \frac{\varepsilon}{T} \sum_{i=1,2} \int_{F_2}^{F_1} \frac{X(x, y, P_i, 0)}{G(x, y, P_i)}\, dy, \tag{6}$$

where

$$T = \sum_{i=1,2} \int_{F_2}^{F_1} \frac{dy}{G(x, y, P_i)}$$

is the period of oscillation,

$$B_i = \left(\frac{\partial P_i}{\partial F_1}\right)^{-1} \left\{-\frac{\partial P_i}{\partial y}\, g(x, y, P_i, 0) + f(x, y, P_i, 0) - \frac{\partial P_i}{\partial x}\, X(x, y, P_i, 0)\right\}.$$

Here $p = P_i(x, y, F_1)$ (by P_1 and P_2 are denoted the branches of the function (4) corresponding to positive and negative values of $\dot{y}$), with $i = 1$ corresponding to the variation of y from F_1 to F_2; $i = 2$, to the variation of y from F_2 to F_1. The perturbations f, g, X are taken at $\varepsilon = 0$, since here we restrict ourselves to consideration of the first approximation.

The phase of the oscillations ψ in the first approximation is found from the equation

$$\dot{\psi} = \frac{2\pi}{T} + \frac{\varepsilon}{T} \sum_{i=1,2} \int_{F_2}^{F_1} \frac{\Psi_i(x, F_1, y)}{G(x, y, P_i)}\, dy, \tag{7}$$

where

$$\Psi_i = 2\pi(-1)^{i+1} \left\{ \frac{-1}{TG(x, y, P_i)}\, g(x, y, P_i, 0) + \right.$$

$$+ \frac{\partial}{\partial F_1} \left(\frac{1}{T} \int_y^{F_1} \frac{dy}{G(x, y, P_i)} \right) B_i(x, y, F_1) +$$

$$\left. + \frac{\partial}{\partial x} \left(\frac{1}{T} \int_y^{F_1} \frac{dy}{G(x, y, P_i)} \right) X(x, y, P_i, 0) \right\}.$$

However, in order to maintain the required accuracy in (7), it is necessary to substitute the values of the slowly varying variables F_1, F_2, x, computed in the second approximation in ε. (We do not write down here the corresponding equations, which differ from (6) by the presence of terms of order ε^2.) Knowledge of the first approximations for F_1, F_2, x, ψ makes it possible to find the first approximations for y and p. The computation of higher approximations is carried out by the same method; we do not give here the formulas for higher approximations.

§ 3. **Some special cases.** Consider an oscillatory system differing from (3) in that the force Q producing the oscillations depends on $\dot{y}$:

$$\frac{d}{dt}\,[m(x)\dot{y}] + Q(x, y, \dot{y}) = \varepsilon f(x, y, \dot{y}, \varepsilon),$$

$$\dot{x} = \varepsilon X(x, y, \dot{y}, \varepsilon). \tag{8}$$

(Some problems for systems that are special cases of (8) were considered in (3).) System (8) is a special form of (2), and therefore all the results of § 2 are applicable also to system (8). As a result, simple computational formulas are obtained; they are not given here.

Let $Q = P(x, y) + R(x, y)\dot{y} + S(x, y)\dot{y}^2$; then (4) is an integral of Abel' s equation of the second kind.

If $Q = P(x, y) \cdot R(x, \dot{y})$, then the integral (4) has the form:

$$\int \frac{p\, dp}{R(x, p/m)} = -m \int P(x, y)\, dy + C. \tag{9}$$

Let $Q = P(x, y) + R(x, y)\dot{y}^2$; then (4) is given by the formula

$$p = \exp\left\{-\frac{1}{m}\int R\,dy\right\}\left(C - 2m\int P\exp\left\{\frac{2}{m}\int R\,dy\right\}dy\right)^{1/2}. \qquad (10)$$

If $Q = R(x, y)\dot{y}^2 + S(x, y)\dot{y}^{2\alpha}$, then for the integral (4) the following expression is valid:

$$p = \exp\left\{-\frac{1}{m}\int R\,dy\right\}\left(C + \frac{2(\alpha-1)}{m^{2\alpha-1}}\int S\exp\left\{\frac{2(1-\alpha)}{m}\int R\,dy\right\}dy\right)^{1/2(1-\alpha)}. \qquad (11)$$

Substitution of formulas (9)–(11) into the equations of § 2 leads to quadrature expressions for the equations of the first and higher approximations.

In conclusion, we note that with the aid of asymptotic methods based on averaging, it is possible to investigate general systems of the form (2) also in the case when the perturbing factors depend explicitly on time (in particular, the analysis of stationary resonant regimes).

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Note: Figure translations are in progress. See original paper for figures.