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Abstract

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COVERING THEOREMS FOR FUNCTIONS REGULAR AND UNIVALENT IN THE DISK

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Let $S(a_1, a_2)$ be the class of functions $f(z) = c_1 z + \cdots$, regular and univalent in the disk $B: \ |z| < 1$ and not assuming in it the prescribed values a_1 and a_2; $S^*(a_1, a_2)$ the subclass of starlike functions from $S(a_1, a_2)$; $f(z; a_1, a_2)$ that function of the class $S(a_1, a_2)$ for which $|f'(0)| \le f'(0; a_1, a_2)$, $f(z) \in S(a_1, a_2)$; $f^*(z; a_1, a_2)$ the analogous extremal function for the class $S^*(a_1, a_2)$.

Let S be the class of functions $f(z) = z + c_1 z^2 + \cdots$, regular and univalent in B; S^* the subclass of starlike functions from S; $S_+(S^*_+)$ the set consisting of the function $f(z) = z$ and all those functions of the class $S(S^*)$ for which, among the coefficients $c_2, c_3, \ldots$, the first coefficient different from zero is positive.

In this paper $\max|f'(0)|$ is found in the classes $S(a_1, a_2)$ and $S^*(a_1, a_2)$, and the functions $f(z; a_1, a_2)$ and $f^*(z; a_1, a_2)$ are determined. With the aid of these results some covering theorems are established in the classes S, S^*, and the largest sets belonging to the image of the disk B under its mapping by any function of the classes S_+ and S^*_+, respectively, are determined.

In what follows, $K(k)$ is the complete elliptic integral of the first kind with modulus k; $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, $\theta(u)$ are Jacobi elliptic functions with the same modulus k; $K'(k) = K(\sqrt{1-k^2})$.

Let $0 < a \le 1$, $0 \le \alpha \le \pi/2$; m, p, ω, k, δ be continuous functions of a and α, defined for $0 < a \le 1$, $0 < \alpha < \pi/2$ by the system

$$p = \sqrt{m^2 - \left(e^{-i\alpha} + \frac{1}{a^2} e^{i\alpha}\right) m + \frac{1}{a^2}}\ {}^*, \qquad k^2 = \frac{p + m - \frac{1}{2}\left(e^{-i\alpha} + \frac{1}{a^2} e^{i\alpha}\right)}{2p},$$

$$\operatorname{cn}\omega = \frac{m-p}{m+p}, \qquad \left(\frac{\sqrt{p/m}}{a(m+p)} - \frac{\theta'(\omega)}{\theta(\omega)}\right) K = i\delta, \qquad 2\delta \operatorname{Im}(iK'/K) = \pi \operatorname{Im}(\omega/K) \tag{1}$$

$$(\omega = \lambda_1 K + \lambda_2 i K', \qquad \text{where } 0 < \lambda_1 < 1, \quad -1 < \lambda_2 < 1);$$

$$h(a,\alpha) = \frac{a^2}{4}\left|\frac{(m+p)^2\theta^2(0)}{m\theta^2(\omega)}\right| e^{\delta\,\mathrm{Im}\,\omega/K}, \qquad 0<\alpha<\frac{\pi}{2};$$

$$h(a,0)=\frac{1}{4}, \qquad h\left(a,\frac{\pi}{2}\right)=\frac{1+a^2}{4};$$

$w = F(z,a,\alpha)$ is determined for $z \in B$, $0 < a \leq 1$, by the conditions

$$w = m - p\,\frac{1+\mathrm{cn}\,u}{1-\mathrm{cn}\,u}, \qquad ze^{-i\psi} = \frac{\mathrm{sn}\,\frac{u-\omega}{2}\;\mathrm{dn}\,\frac{u-\omega}{2}\;\theta^2\left(\frac{u-\omega}{2}\right)}{\mathrm{sn}\,\frac{u+\omega}{2}\;\mathrm{dn}\,\frac{u+\omega}{2}\;\theta^2\left(\frac{u+\omega}{2}\right)} e^{-i\delta u/K}\;{}^{**}.$$

$$\psi = \arg\left\{\frac{m\theta^2(\omega)}{(m+p)^2\theta^2(0)} e^{i\delta\omega/K}\right\}, \qquad 0<\alpha<\frac{\pi}{2};$$

$$F(z,a,0) = \frac{4z}{(1+z)^2}, \qquad F\left(z,a,\frac{\pi}{2}\right) = \frac{4}{1+a^2}\,\frac{z}{1+2i\frac{1-a^2}{1+a^2}z - z^2}.$$

* Here and below that branch of the square root is considered whose values are positive for positive values of

** By the function $u = u(z,a,\alpha)$ is meant that branch for which $u(0,a,\alpha) = \omega(a,\alpha)$.

Theorem 1. In the class $S\left(e^{-i\alpha}, \frac{1}{a^2}e^{i\alpha}\right)$, $0 < a \leqslant 1$, $0 \leqslant \alpha \leqslant \pi/2$, of functions $f(z) = c_1 z + \ldots$, the sharp inequality

$$|c_1| \leqslant \frac{1}{h(a,\alpha)}. \tag{2}$$

holds.

Equality in (2) occurs only for the function

$$f\left(\varepsilon z; e^{-i\alpha}, \frac{1}{a^2}e^{i\alpha}\right), \quad |\varepsilon| = 1,$$

where

$$f\left(z; e^{-i\alpha}, \frac{1}{a^2}e^{i\alpha}\right) = F(z,a,\alpha).$$

For $0 < \alpha < \frac{\pi}{2}$, each of the extremal functions maps the disk B onto the whole w-plane with a single slit consisting of three analytic arcs, two of which join, respectively, the points $e^{-i\alpha}$, m and $\frac{1}{a^2}e^{i\alpha}$, m, while the third has its origin at the point m and goes to infinity. For $\alpha = 0$ and $\alpha = \pi/2$, the extremal functions map the disk B onto the whole w-plane with a single radial slit $w \geqslant 1$ in the first case, and with two radial slits $w = -it$, $t \geqslant 1$, and $w = it$, $t \geqslant 1/a^2$, in the second case.

The proof of the theorem uses known properties of the function $w = f(z)$ realizing $\max|f'(0)|$ in the given class, in particular the uniqueness of the extremal function normalized by the condition $f'(0) > 0$, and the differential equation for the extremal function (1). In the limiting cases $\alpha = 0$ and $\alpha = \pi/2$, we obtain the known results of Koebe and M. A. Lavrent' ev for the mapping considered in the theorem.

Using the simple relation between the class $S(a_1, a_2)$, where a_1, a_2 are arbitrary prescribed values, and the class

$$S\left(e^{-i\alpha}, \frac{1}{a^2}e^{i\alpha}\right)$$

for suitable a and α, from Theorem 1 we immediately obtain

Corollary 1. In the class $S(a_1, a_2)$, $0 < |a_1| \leqslant |a_2|$, of functions $f(z) = c_1 z + \ldots$, the inequality holds ($a^2 = |a_1/a_2|$, $\alpha = \arg\sqrt{a_1/a_2}$, $-\pi/2 < \alpha \leqslant \pi/2$)

$$|c_1| \leqslant \frac{|a_1|}{h(a, |\alpha|)}. \tag{3}$$

Equality in (3) is realized only by the function $f(\varepsilon z; a_1, a_2)$,

$$|\varepsilon| = 1, \quad f(z; a_1, a_2) = a_1 e^{i\alpha} f\left(\frac{|a_1|}{a_1} e^{-i\alpha} z;\ e^{-i\alpha},\ \frac{1}{a^2}e^{i\alpha}\right)$$

$$\left(f(z; \overline{a_1}, \overline{a_2}) = \overline{f}(z; a_1, a_2)\right)^*.$$

Corollary 2. Let a, $0 < a \leqslant 1$, and α, $-\pi/2 < \alpha \leqslant \pi/2$, be prescribed numbers, and let $f(z) \in S$ not take in the disk B the values a_1 and a_2, for which $a_1/a_2 = a^2 e^{2i\alpha}$. Then the inequality holds

$$|a_1| \geqslant h(a, |\alpha|). \tag{4}$$

Equality in (4) occurs only for functions

$$f\left(z; a_1, \frac{1}{a^2}e^{-i\alpha}a_1\right),$$

where $a_1 = h(a, \alpha)e^{i\beta}$, and β is an arbitrary real number.

For $a = 1$, from Theorem 1 and its corollaries we obtain, as a special case, the results stated in ([2]).

Remark. From the differential equation for the function

$$f\left(z; e^{-i\alpha}, \frac{1}{a^2}e^{i\alpha}\right) = c_1(a,\alpha)z + c_2(a,\alpha)z^2 + \dots$$

([1]), and from the system (1), we find that, for $0 < a \leqslant 1$, $0 \leqslant |\alpha| < \pi/2$, there is defined a continuous function

$$\gamma(a,\alpha) = \arg c_2(a,\alpha), \qquad \gamma(1,\alpha) = \pi \quad \text{for} \quad 0 \leqslant |\alpha| < \pi/2.$$

Let $f(B)$ be the image of the disk B in the w-plane under the mapping $w = f(z)$; let $\overline{f}(B)$ be the domain symmetric to $f(B)$ with respect to the real axis.

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* The function $\overline{f}(z)$ is obtained from the function $f(z) = c_1 z + \dots + c_{nz}^n + \dots$ by replacing all its coefficients c_n by $\overline{c}_n$.

Set

$$D(S) = \bigcap_{f\in S} f(B), \qquad \mathfrak{D}(S) = \bigcap_{f\in S} [f(B) \cup \overline{f(B)}];$$

the corresponding sets for subclasses of S are denoted analogously.

From the results obtained from Theorem 1 and its corollaries for $a = 1$, and presented in (2), one easily finds the set $\mathfrak{D}(S)$, which coincides with the largest set belonging to the image of the disk B under its mapping by any function of the class S with real coefficients, and all its boundary functions (2) are determined.

It is well known (and follows in an obvious way from Corollary 2 to Theorem 1) that the set $D(S)$ is the disk $|w| < 1/4$, and to each boundary point $w_\theta = \frac{1}{4}e^{i\theta}$ of it there corresponds only the function

$$f_\theta(z) = \frac{z}{(1 + e^{-i\theta}z)^2}.$$

To obtain a strengthening of this result of Koebe, it is natural to consider the class S_+ (3). Clearly, the set $D(S_+)$ is symmetric with respect to the real axis, and it follows from simple considerations that the part of the boundary $w = R_+(\varphi)e^{i\varphi}$ (R_+, φ are polar coordinates) of this set lying in the half-plane $\mathrm{Re}\, w \geq 0$ is the semicircle $w = \frac{1}{2}e^{i\varphi}$, $|\varphi| \leq \pi/2$ (with all interior points of this semicircle belonging to the set under consideration), $R_+(\pi) = 1/4$ (3). With the aid of Theorem 1 and the results following from it, the function $R_+(\varphi)$ is determined in principle for $0 < |\varphi - \pi| < \pi/2$. This gives the set $D(S_+)$.

Let φ_0 be fixed, $0 < \varphi_0 < \pi/2$; let $\rho(\varphi_0)$ be the least value of the function $h(a, |\alpha|)$, $0 < a \leq 1$, $0 < |\alpha| < \pi/2$, under the condition

$$\gamma(a,\alpha) - \alpha = \pi - \varphi_0.$$

Theorem 2. *The set $D(S_+)$ is bounded by the curve $w = R_+(\varphi)e^{i\varphi}$, where*

$$R_+(\varphi) = \frac{1}{2}, \quad |\varphi| \le \pi/2; \qquad R_+(\varphi) = \rho(|\pi - \varphi|), \quad 0 < |\pi - \varphi| < \pi/2;$$

$$R_+(\pi) = \frac{1}{4}.$$

The semicircle $w = \frac{1}{2}e^{i\varphi}$, $-\pi/2 < \varphi < \pi/2$, belongs to the set $D(S_+)$; the remaining boundary points of the set $D(S_+)$ do not belong to it.

We now give the solution of the same problems for starlike functions.

Let, for $0 < a \le 1,\ 0 \le \lambda \le 1,\ \psi = \psi(a, \lambda)\ (0 \le \psi \le \pi/2)$ be the solution of the equation

$$g(\psi, \lambda) = \left(\frac{1 + \lambda \cos^2 \psi - \sin \psi \sqrt{1 - \lambda^2 \cos^2 \psi}}{1 + \lambda \cos^2 \psi + \sin \psi \sqrt{1 - \lambda^2 \cos^2 \psi}} \right)^{(1+\lambda)/2} \times$$

$$\times \left(\frac{1 - \lambda \cos^2 \psi - \sin \psi \sqrt{1 - \lambda^2 \cos^2 \psi}}{1 - \lambda \cos^2 \psi + \sin \psi \sqrt{1 - \lambda^2 \cos^2 \psi}} \right)^{(1-\lambda)/2} = a^2.$$

Let $0 < a \le 1,\ 0 \le \alpha \le \frac{\pi}{2}$; $\lambda = 1 - \frac{2}{\pi}\alpha$;

$$h^*(a, \alpha) = \frac{1}{2} \left(1 + \lambda \cos^2 \psi + \sin \psi \sqrt{1 - \lambda^2 \cos^2 \psi}\right)^{-(1+\lambda)/2} \times$$

$$\times \left(1 - \lambda \cos^2 \psi + \sin \psi \sqrt{1 - \lambda^2 \cos^2 \psi}\right)^{-(1-\lambda)/2}, \qquad 0 < \alpha < \frac{\pi}{2};$$

$$h^*(a, 0) = \frac{1}{4}, \qquad h^*\left(a, \frac{\pi}{2}\right) = \frac{1 + a^2}{4};$$

$$F^*(z, a, \alpha) = \frac{1}{h^*(a, \alpha)} \frac{z}{(1 + e^{i(1-\lambda)\psi} z)^{1+\lambda} (1 - e^{-i(1+\lambda)\psi})^{1-\lambda}}, \qquad |z| < 1^*.$$

The following is based directly on Lindelöf' s principle.

Theorem 3. *In the class $S^*(a_1, a_2)$, $0 < |a_1| \le |a_2|$, of functions $f(z) = c_1 z + ...$, the inequality*

$$\left(a^2 = \frac{a_1}{a^2}, \quad d = \arg \sqrt{\frac{a_1}{a^2}}, \quad -\frac{\pi}{2} < \alpha \le \frac{\pi}{2}\right) \qquad |c_1| \le \frac{|a_1|}{h^*(a, |\alpha|)}. \qquad (5)$$

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* Here and below, by the function $F^*(z, a, \alpha)$ is meant that branch of it for which

$$\frac{h^*(a, \alpha) F^*(z, a, \alpha)}{z} \to 1 \quad \text{as } z \to 0.$$

The equality sign in (5) occurs only for the functions $f^*(\varepsilon z;\, a_1, a_2)$, $|\varepsilon| = 1$,

$$f^*(z;\, a_1, a_2) = a_1 e^{i\alpha} F^* \left(\frac{|a_1|}{a_1} e^{-i\alpha} z,\, a,\, \alpha \right) \quad \text{for } 0 \le \alpha \le \pi/2,$$

$$f^*(z;\, \bar a_1, \bar a_2) = \overline{f^*(z;\, a_1, a_2)}.$$

For $0 < |\alpha| < \pi/2$, each of the extremal functions maps the disk B onto the whole w-plane with two radial slits $\arg w = \arg a_1$, $|w| \ge |a_1|$, and $\arg w = \arg a_2$, $|w| \ge |a_2|$; for $\alpha = 0$, onto the whole w-plane with a single radial slit $\arg w = \arg a_1$, $|w| \ge |a_1|$.

Let, for $0 < \varphi_0 < \pi/2$, $\lambda_0 = \lambda(\varphi_0)$, $\chi_0 = \chi(\varphi_0)$ $(0 < \lambda_0 < 1,\ 0 < \chi_0 < \pi/2)$ be the solution of the system

$$\ln \frac{1+\lambda}{\sqrt{1-\lambda^2 \sin^2 \chi} + \lambda \cos \chi} - \frac{\sin \chi}{\sqrt{1-\lambda^2 \sin^2 \chi} + \cos \chi} \, \frac{\chi(1+\lambda^2 \operatorname{tg}^2 \chi) - \operatorname{tg} \chi}{\lambda \operatorname{tg}^2 \chi} = 0,$$

$$\lambda \operatorname{tg} \chi = \operatorname{tg}(\varphi_0 + \lambda \chi);$$

$$\rho^*(\varphi_0) = \frac{1}{2} \left(1 + \lambda_0 \sin^2 \chi_0 + \cos \chi_0 \sqrt{1-\lambda^2 \sin^2 \chi_0}\right)^{-(1+\lambda_0)/2} \left(1 - \lambda_0 \sin^2 \chi_0 + \cos \chi_0 \sqrt{1-\lambda_0^2 \sin^2 \chi_0}\right)^{-(1-\lambda_0)/2}$$

Theorem 4. The set $D(S_+^*)$ is bounded by the curve $w = R_+^*(\varphi) e^{i\varphi}$, where $R_+^*(\varphi) = 1/2$, $|\varphi| \le \pi/2$; $R_+^*(\varphi) = \rho^*(|\pi - \varphi|)$, $0 < |\pi - \varphi| < \pi/2$; $R_+^*(\pi) = 1/4$.

The semicircle $w = \frac{1}{2} e^{i\varphi}$, $-\pi/2 < \varphi < \pi/2$, belongs to the set $D(S_+^*)$; the remaining boundary points of this set do not belong to it. The point $w_\pi = -1/4$ does not belong to the domain $f(B)$, where $f \in S_+^*$, only for the function $f(z) = z/(1-z)^2$; the points $w_{-\pi/2} = -i/2$ and $w_{\pi/2} = i/2$ only for the function $f(z) = z/(1-z^2)$; the point $w_\varphi = R_+^*(\varphi) e^{i\varphi}$ for $\pi/2 < \varphi < \pi$ only for the function

$$f^* \left(z;\, R_+^* e^{i\varphi},\, \frac{1}{\mu^2} R_+^* e^{i[\varphi + (1-\lambda_0)\pi]} \right),$$

for $\pi < \varphi < 3\pi/2$ only for the function

$$f^* \left(z;\, R_+^* e^{i\varphi},\, \frac{1}{\mu^2} R_+^* e^{i[\varphi - (1-\lambda_0)\pi]} \right),$$

where $\mu = g(\pi/2 - \chi_0, \lambda_0)$, $\lambda_0 = \lambda_0(|\pi - \varphi|)$, $\chi_0 = \chi_0(|\pi - \varphi|)$.

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REFERENCES

1. M. A. Lavrent' ev, *Tr. Fiz.-matem. inst. im. V. A. Steklova AN SSSR*, **5**, 159 (1934).
2. G. V. Kuz' mina, *DAN*, **142**, No. 1, 29 (1962).
3. W. T. Scott, *Am. Math. Monthly*, **64**, No. 8, pt. 2, 90 (1957).

Note: Figure translations are in progress. See original paper for figures.