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A. A. BUKHSHTAB

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Abstract

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MATHEMATICS

A. A. BUKHSHTAB

NEW RESULTS

IN THE STUDY OF THE GOLDBACH-EULER PROBLEM

AND THE TWIN PRIME PROBLEM

(Presented by Academician Yu. V. Linnik on 8 XII 1964)

The Goldbach–Euler problem on representing even numbers as sums of two prime numbers has not yet been solved. The method of Eratosthenes' sieve, in combination with methods developed by Yu. V. Linnik and his followers in the theory of Dirichlet L-series, made it possible to find such k that all sufficiently large numbers $2N$ are representable in the form $2N = p + n$, where p is prime and n consists of no more than k prime factors. The existence of such a k was first proved by A. Rényi ([1]). The value $k = 4$ was obtained in works of B. V. Levin, M. B. Barban, Wang Yuan, and Pan Cheng-dong ([2–4]). Corresponding results were also obtained in the problem of the existence of an infinite set of prime numbers p such that $p + 2$ consists of no more than k prime factors. In the present work I have obtained for k the value equal to 3.

Theorem 1. There exists N_0 such that every even number greater than N_0 is representable as the sum of a prime number and a number having no more than three prime factors.

Theorem 2. There exists an infinite set of prime numbers p such that $p + 2$ is the product of no more than three prime factors.

In the proof, an essential role is played by the application of the following theorem of M. B. Barban ([4]).

Theorem A. Let ν be any number less than ${}^3/_8$; let A be an arbitrary constant number; then

$$\sum_{D\le x^\nu}\mu^2(D)\max_{\substack{a \pmod D\\(a,D)=1}}\left|\pi_a(x,D)-\frac{\mathrm{li}(x)}{\varphi(D)}\right| = O\left(\frac{x}{\ln^A x}\right).$$

Here $\pi_a(x,D)$ is the number of primes $p\le x$ such that $p\equiv a \pmod D$; $\varphi(D)$ is Euler' s function, $\mu(D)$ is the Möbius function.

Theorem 2 is proved. The transition to the proof of Theorem 1 is carried out in the usual way.

Take a natural number q and prime numbers $2<p_1<p_2<\dots<p_r<z\le p_{r+1}$, $p_i\nmid q$. To q associate a number a, and to each p_i a number a_i, so that $(a,q)=1$, $p_i\nmid a_i$. The selected set $a,a_1,\dots,a_r$ will, for brevity, be denoted by the letter ω. Denote by $P_\omega(x,q,z)$ the number of primes $p\le x$ such that $p\equiv a \pmod q$, $p\not\equiv a_i \pmod{p_i}$ for $1\le i\le r$. By the method of Vigo Brun it is proved

Theorem B. There exist nondecreasing functions $\lambda(\alpha)$ and $\Lambda(\alpha)$ such that, for $\alpha>0$, $q<x^\nu$,

$$P_\omega\left(x,q,\left(\frac{x^\nu}{q}\right)^{1/\alpha}\right)>$$

$$>\left\{B_0\lambda(\alpha)+O\left(\frac{1}{(\nu\ln x-\ln q)^{1/2}}\right)\right\}\frac{c(q)\,\mathrm{li}(x)}{\nu\ln x-\ln q}-r_\omega\left(x,q,\left(\frac{x^\nu}{q}\right)^{1/\alpha}\right), \quad (1)$$

$$P_\omega\left(x,q,\left(\frac{x^\nu}{q}\right)^{1/\alpha}\right)<$$

$$<\left\{B_0\Lambda(\alpha)+O\left(\frac{1}{(\nu\ln x-\ln q)^{1/2}}\right)\right\}\frac{c(q)\,\mathrm{li}(x)}{\nu\ln x-\ln q}+r_\omega\left(x,q,\left(\frac{x^\nu}{q}\right)^{1/\alpha}\right),$$

where B_0 is a constant independent of the choice of the domain ω; $\lambda(\alpha)>0$ for $\alpha\ge 10$,

$$r_\omega\left(x,q,\left(\frac{x^\nu}{q}\right)^{1/\alpha}\right)<\sum_{D\in\Omega}\mu^2(D)\max_{\substack{a\pmod D\\(a,D)=1}}\left|\pi_a(x,D)-\frac{\mathrm{li}(x)}{\varphi(D)}\right|. \quad (2)$$

The domain $\Omega=\Omega\left(x;q,(x^\nu/q)^{1/\alpha}\right)$ of values D consists of numbers D such that $D=qm$, $m<x^\nu/q$, and the greatest prime divisor of m is less than $(x^\nu/q)^{1/\alpha}$;

$$c(q) = \frac{1}{\varphi(q)} \prod_{\substack{p/q \\ p\neq 2}} \frac{p-1}{p-2}.$$

In the course of the proof, explicit formulas were obtained for $\lambda(\alpha)$ and $\Lambda(\alpha)$. The values of the function $\Lambda(\alpha)$ for $\alpha \leq 10$ with step 0.01, and $\lambda(10) = 9.999942$, were computed on a Minsk-1 computer at the Moscow State Pedagogical Institute named after V. I. Lenin. The values obtained defined the step functions $\Lambda_0(\alpha)$ and $\lambda_0(\alpha)$, with $\lambda_0(\alpha) = 0$ for $\alpha < 10$. Applying, as in Wang Yuan' s work, the method of A. A. Buchstab, one can prove the following theorem:

Theorem B. Let $\beta > 1$. Inequalities (1) remain valid if in them $\lambda(\alpha)$ and $\Lambda(\alpha)$ are replaced respectively by

$$\bar{\lambda}(\alpha) = \begin{cases} \max\left(\lambda(\alpha),\, \lambda(\beta) - \int_{\alpha-1}^{\beta-1} \frac{\Lambda(z)}{z}\, dz\right), & \text{for } 1 < \alpha \leq \beta, \\ \lambda(\alpha), & \text{for } 0 < \alpha \leq 1 \text{ or } \alpha > \beta; \end{cases} \tag{3}$$

$$\bar{\Lambda}(\alpha) = \begin{cases} \min\left(\Lambda(\alpha),\, \Lambda(\beta) - \int_{\alpha-1}^{\beta-1} \frac{\lambda(z)}{z}\, dz\right), & \text{for } 1 < \alpha \leq \beta, \\ \Lambda(\alpha), & \text{for } 0 < \alpha \leq 1 \text{ or } \alpha > \beta. \end{cases}$$

After such a replacement, the remainder terms will still satisfy condition (2) with the same domain Ω.

Starting from the functions $\lambda_0(\alpha)$ and $\Lambda_0(\alpha)$ defined above, with the aid of inequalities (3), for values $0 < \alpha \leq 10$ the functions

$$\lambda_0(\alpha) \leq \lambda_1(\alpha) \leq \lambda_2(\alpha) \leq \cdots \leq \Lambda_2(\alpha) \leq \Lambda_1(\alpha) \leq \Lambda_0(\alpha).$$

were constructed.

Successive computation on the same computer of the functions $\lambda_i(\alpha)$ with deficiency and $\Lambda_i(\alpha)$ with excess yielded a table of values of $\lambda(\alpha)$ and $\Lambda(\alpha)$ (indices omitted for simplicity of notation), in which, in particular, the following data were obtained:

α	$\alpha \leq 3$	3.1	3.2	3.3	3.4	3.5	3.6
$\Lambda(\alpha)$	3.580161	3.58619	3.60711	3.64053	3.68437	3.73696	3.79694

α	3.7	3.8	3.9	4	4.1	4.2	4.3
$\Lambda(\alpha)$	3.86318	3.93473	4.01079	4.09072	4.17392	4.25994	4.34834

α	4.4	4.5	4.6	4.7	4.8	4.9	5
$\Lambda(\alpha)$	4.43877	4.53094	4.62455	4.71940	4.81526	4.91197	5.00938

The method used in the proof of Theorem B also makes it possible to prove the following theorems:

Theorem G. *Let* $3/8\nu < \alpha \leq \beta,\ \nu_1 < \nu$; *then*

$$\sum_{x^{3/3\beta} \leq p < x^{3/8\alpha}} P_\omega(x, p, p) < \frac{B_0}{\nu_1} \frac{\mathrm{li}(x)}{\ln x} \int_{\alpha-1}^{\beta-1} \frac{\Lambda(z)}{z}\, dz + O\left(\frac{x}{\ln^{5/2} x}\right). \tag{4}$$

Theorem D. *Let* $3/8\nu < \alpha \leq \beta \leq \delta,\ \nu_1 < \nu$; *then*

$$\sum_{x^{3/8\beta} \leq p < x^{3/8\alpha}} P_\omega(x, p, x^{3/8\delta}) < \frac{B_0}{\nu_1} \frac{\mathrm{li}(x)}{\ln x} \int_{\alpha-1}^{\beta-1} \Lambda\left(\frac{\delta z}{z+1}\right) \frac{dz}{z} + O\left(\frac{x}{\ln^{5/2} x}\right). \tag{5}$$

Consider the intervals $I_n = [x^{n/64}, x^{(22-n)/64})$ for $4 \leq n \leq 10$; $I_n = [x^{n/64}, x^{(n+1)/64})$ for $18 \leq n \leq 20$, and $L_n = [x^{n/64}, x^{(n+1)/64})$ for $4 \leq n \leq 10$. To the intervals I_n we assign the numbers c_n, setting $c_4 = 4/21,\ c_n = 1/21$ for $5 \leq n \leq 10$, $c_n = (21-n)/n$ for $18 \leq n \leq 20$, and to the intervals L_n the numbers $d_n = (21-2n)/21$. We consider the function $P(x, q, z) = P_\omega(x, q, z)$ in the particular case when all the numbers a and a_i constituting ω are taken equal to -2.

Theorem E. *Let* $\mathfrak{G}(x)$ *denote the number of primes* $p < x - 2$ *such that: 1)* $p + 2 \not\equiv 0 \pmod{p_i}$ *for all* $p_i < x^{1/16}$*; 2)* $p + 2$ *contains at least four distinct prime factors. The set of such* p *will henceforth be denoted by* $\mathfrak{G}$.

Define $S(x)$ by the equality

$$S(x) = \sum_{4 \leq n \leq 10} c_n \sum_{p_i \in I_n} P(x, p_i, x^{n/64}) + \sum_{18 \leq n \leq 20} c_n \sum_{p_i \in I_n} P(x, p_i, x^{1/16}) + \\ + \sum_{4 \leq n \leq 10} d_n \sum_{p_i \in L_n} P(x, p_i, p_i). \tag{6}$$

Then $\mathfrak{G}(x) \leq S(x)$.

In the proof, by M is denoted the set of primes $p < x-2$ such that $p+2 \not\equiv 0 \pmod{p_i}$ for all $p_i < x^{1/16}$. Each $p+2$, where $p \in \mathfrak{G}$ ($\mathfrak{G} \subset M$), is written in the form $p+2 = p_\alpha^{(k_1)} p_\beta^{(k_2)} p_\gamma^{(k_3)} p_\delta^{(k_4)} m$, where $p_\alpha^{(k_1)} < p_\beta^{(k_2)} < p_\gamma^{(k_3)} < p_\delta^{(k_4)}$ are the four smallest distinct prime divisors of $p+2$. The notation $p^{(t)}$ for $4 \le t \le 20$ means that $x^{t/64} \le p^{(t)} < x^{(t+1)/64}$, while for $t = 21$ it means that $x^{21/64} \le p^{(21)} < x$. $S(x)$ is written in the form $S(x) = \sum_{p \in M} T(p)$, where

$$T(p) = \sum_{p_i | p+2} \sum_{\substack{4 \le n \le 10 \\ p_i \in I_n,\ p \in M_n}} c_n + \sum_{p_i | p+2} \sum_{\substack{18 \le n \le 20 \\ p_i \in I_n}} c_n + d(p).$$

$p \in M_n$ means that $p+2 \not\equiv 0 \pmod{p_i}$ for all $p_i < x^{n/64}$; $d(p) = d_n$ if $p_\alpha^{(k_1)} \in L_n$ ($4 \le n \le 10$), and $d(p) = 0$ if $p_\alpha^{(k_1)} \notin L_n$ for all such n.

Selecting, for $p \in \mathfrak{G}$, in $T(p)$ the sum of those c_n and $d_n = d(p)$ which correspond to the values p_i that are prime divisors of $p_\alpha^{(k_1)} p_\beta^{(k_2)} p_\gamma^{(k_3)} p_\delta^{(k_4)}$, we obtain a quantity $U(p) \le T(p)$. To prove the theorem it is enough to show that $U(p) \ge 1$ for every $p \in \mathfrak{G}$. Indeed, then

$$S(x) \ge \sum_{p \in \mathfrak{G}} T(p) \ge \sum_{p \in \mathfrak{G}} U(p) \ge \sum_{p \in \mathfrak{G}} 1 = \mathfrak{G}(x).$$

In proving that $U(p) \ge 1$ for all $p \in \mathfrak{G}$, one has, considering all possible values k_1, k_2, k_3, k_4, to estimate this function 108 times; let us note that in all these 108 cases there exist such k_1, k_2, k_3, k_4, i.e. such $p \in \mathfrak{G}$, that $U(p) = 1$.

The problem of optimally choosing the quantities c_n and d_n is, in essence, a problem of linear programming, namely that of finding the minimum of the linear form (6) under the condition that the inequalities $U(p) \geqslant 1$, linear with respect to c_n and d_n, are satisfied.

In (6), $S(x)$ is written as the sum of 10 terms of the form

$$c_n \sum_{p_i \in I_n} P(x, p_i, x^{s/64})$$

and 7 terms of the form

$$d_n \sum_{p_i \in L_n} P(x, p_i, p_i).$$

Such sums are estimated with the aid of theorems Γ and Δ and of the table of values of $\Lambda(a)$. Taking $v_1 = 3/8 - 1/10^7$, we obtain that, for $x > x_0$,

$$\mathfrak{S}(x) \leqslant S(x) < 15.0607 B_0 x / \ln^2 x.$$

Denote by $P(x)$ the number of primes $p \leqslant x$ such that $p+2$ is not divisible by prime numbers smaller than $x^{1/16}$. Taking $a = a_1 = \dots = a_r = -2$, we have, for $x > x_0$,

$$P(x) = P_\omega(x, 1, x^{1/16}) > \frac{8}{3} B_0 \Lambda(6) x / \ln^2 x > 15.9979 B_0 x / \ln^2 x.$$

For the number $K(x)$ of primes $p \leqslant x$ such that $p+2$ is square-free and has no prime divisors smaller than $x^{1/16}$, we have, for $x > x_0$,

$$K(x) < 0.0001 B_0 x / \ln^2 x.$$

Denote by $F(x)$ the number of primes $p \leqslant x$ such that: 1) $p+2$ has no prime divisors smaller than $x^{1/16}$; 2) $p+2$ is square-free; 3) $p+2$ has no more than three prime factors. Then, for $x > x_0$,

$$F(x) \geqslant P(x) - \mathfrak{S}(x) - K(x) - 2 > 0.937 B_0 x / \ln^2 x.$$

$F(x) \to \infty$ as x increases, so that Theorem 2 is proved. To prove Theorem 1, in the definitions of $\mathfrak{S}(x)$, M, $P(x)$, $K(x)$, $F(x)$, one must replace $p+2$ by $2N - p$. Accordingly, in the set $p_1, \dots, p_r$, the divisors of $2N$ are omitted, and in $P(x, p_i, x^{n/64})$ and $P(x, p_i, p_i)$ one takes $a_i = 2N$.

Moscow State
Pedagogical Institute
named after V. I. Lenin

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Note: Figure translations are in progress. See original paper for figures.