ON TOPOLOGICAL EMBEDDINGS OF POLYHEDRA IN EUCLIDEAN SPACES

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Abstract

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ON TOPOLOGICAL EMBEDDINGS OF POLYHEDRA IN EUCLIDEAN SPACES

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1. This note is a continuation of ([1]). It contains an outline of the proof of one of the results announced in ([2]), concerning topological embeddings of polyhedra. In addition, some closely related results are formulated.

The following note will be devoted to embeddings of manifolds with isolated singular points and to the theorem on the union of locally flat cells (see ([2]) and item 5 of the present paper).

Theorem 1. *Let Q be a compact polyhedron of dimension $k < {}^2/_3 n - 1$, and let $q : Q \to E^n$ be a topological embedding of Q in n-dimensional Euclidean space. Suppose that q is locally flat on every open simplex of some triangulation T of the polyhedron Q. Then for every $\varepsilon > 0$ there exists an ε-homeomorphism $h : E^n \to E^n$, identical outside the ε-neighborhood of Q, such that the homeomorphism $hq : Q \to E^n$ is piecewise linear in the triangulation T.*

The embedding q is piecewise linear on the 0-skeleton of T. Arguing by induction, we may assume that q is already piecewise linear on the $(p-1)$-skeleton of T. The proof of the theorem splits into two steps. First one constructs a homeomorphism $h_1 : E^n \to E^n$ so that $h_1 q$ is piecewise linear on a neighborhood of $|T^{p-1}|$ in $|T^p|$ (where $|T^r|$ is the body of the r-skeleton of the triangulation), and then one constructs a homeomorphism h_2 so that $h_2 h_1 q$ is piecewise linear on $|T^p|$. Number the p-simplices of T and suppose that q is already piecewise linear on a neighborhood of the boundary in each of some number (possibly zero) of the first p-simplices. Denoting by P the image of the union of these neighborhoods and of the $(p-1)$-skeleton of T, we see that the existence of h_1 follows from the following proposition.

Proposition A. *Let P be a p-dimensional polyhedron in E^n, where $p < {}^2/_3 n - 1$, and let $q : \Delta \to E^n$ be an embedding of a p-dimensional simplex in E^n, locally flat in the interior of Δ, piecewise linear on the boundary, with $q\dot{\Delta} \subset P$, $q(\operatorname{Int}\Delta) \cap P = \Lambda$. Then for every $\varepsilon > 0$ there exists an ε-homeomorphism $h : E^n \to E^n$, identical on P and outside the ε-neighborhood of $q\dot{\Delta}$, such that hq is piecewise linear on a neighborhood of $\dot{\Delta}$ in Δ.*

2. Lemma. *Let P be a p-dimensional polyhedron in a piecewise linear sphere S^n, where $p < {}^2/_3 n - 1$, and let $\varphi : \Delta^p \to S^n$ be a continuous mapping of a p-dimensional piecewise linear cell into S^n, piecewise linear and homeomorphic on the boundary $\dot{\Delta}^p$, and such that $\varphi\dot{\Delta}^p = S^{p-1} \subset P$, $\varphi(\operatorname{Int}\Delta^p) \cap P = \Lambda$. Then there exists a homotopy $\varphi_t : \Delta^p \to S^n$ such that $\varphi_0 = \varphi$, φ_1 is piecewise linear and homeomorphic, and for all t, $\varphi_t/\dot{\Delta}^p = \varphi/\dot{\Delta}^p$, $\varphi_t(\operatorname{Int}\Delta^p) \cap P = \Lambda$.*

The proof of the lemma proceeds by induction on p for fixed codimension $c = n - p$ of the polyhedron P in S^n. If $p = 0$, then $\varphi\Delta^p$ is a point outside P, and the lemma is satisfied for $\varphi_t \equiv \varphi$. Suppose that for dimensions smaller than p it has already been proved. With the aid of this inductive assumption we first construct a homotopy $\hat{\varphi}_t$ so that $\hat{\varphi}_1$ is piecewise linear and homeomorphic on a neighborhood of $\dot{\Delta}^p$ in Δ^p. After this the technique of Zeeman is applied.

One may suppose that φ establishes an isomorphism of the triangulations Δ^p and S^{p-1}, that P and S^{p-1} are covered by subcomplexes of the triangulation S^n, and that the triangulation Δ^p has been chosen so that, for some ordering of the vertices of the boundary $a_1, \dots, a_s$, for each j, $1 \le j \le s$, $\Delta^p \setminus \bigcup_{i=1}^{j} \operatorname{St}_{\Delta^p} a_i$ is a piecewise linear cell. Then, in order to construct φ_t, it is enough to prove the following.

Proposition B. *If $\varphi : \Delta^p \to S^n$ satisfies the conditions of the lemma and a is a vertex of $\dot{\Delta}^p$, then there exists a homotopy $\varphi'_t : \Delta^p \to S^n$ such that $\varphi'_0 = \varphi$, φ'_1 is piecewise linear and homeomorphic on $\operatorname{St}_{\Delta^p} a$, $\varphi'_t/\dot{\Delta}^p = \varphi/\dot{\Delta}^p$, and $\varphi'_t(\operatorname{Int}\Delta^p) \cap P = \Lambda$ for all t, $\varphi'_1(\Delta^p \setminus \operatorname{St}_{\Delta^p} a) \cap \varphi'_1(\operatorname{St}_{\Delta^p} a) = \Lambda$.*

Indeed, we can apply this construction successively to all vertices a_i, replacing after the j-th step Δ^p by $\Delta^p \setminus \bigcup_{i=1}^{j} \operatorname{St}_{\Delta^p} a_i$, P–by $P \cup \varphi_1^j(\bigcup \operatorname{St}_{\Delta^p} a_i)$, and φ by $\varphi_1^j/\Delta^p \setminus \bigcup_{i=1}^{j} \operatorname{St}_{\Delta^p} a_i$, where φ_t^j is the homotopy constructed at the j-th step. As $\hat{\varphi}_t$ one may take the superposition of the homotopies $\varphi_t^s \cdots \varphi_t^1$, assuming that φ_t^{j+1}, for all t, is equal to φ_1^j on $\bigcup_{i=1}^{j} \operatorname{St}_{\Delta^p} a_i$.

For the construction of φ'_t we introduce in $\operatorname{St}_{\Delta^p} a$ and in $\operatorname{St}_{S^n} \bar{a}$, $\bar{a} = \varphi a$, the parameter $\alpha = \alpha(x)$, equal to the ratio in which the point x of the star divides the ray of the star from its vertex to its boundary. Denote by Δ_α^{p-1}, S_α^{n-1}, P_α^{p-1} the sets of points with one value of the parameter α, respectively in $\operatorname{St}_{\Delta^p} a$, $\operatorname{St}_{S^n} \bar{a}$, $\operatorname{St}_P \bar{a}$. For $0 < \alpha \le 1$ these sets have triangulations naturally isomorphic to the boundary of the corresponding star. It is easy to arrange that $\varphi(\operatorname{St}_{\Delta^p} a) \subset \operatorname{St}_{S^n} \bar{a}$. Denote now by π_t and $\bar{\pi}_t$ the deformations of $\operatorname{St}_{\Delta^p} a \setminus a$ and $\operatorname{St}_{S^n} \bar{a} \setminus \bar{a}$ onto the boundary of the star along the rays of the star. Then $\pi = \pi_1$ and $\bar{\pi} = \bar{\pi}_1$ are retractions onto the boundary. Put $\breve{\varphi} : \Delta_1^{p-1} \to S_1^{n-1}$ equal to $\bar{\pi}\varphi/\Delta_1^{p-1}$. We are in the conditions of the inductive hypothesis of the lemma, where $S^n, P, \Delta^p, \varphi$ are replaced by $S_1^{n-1}, P_1^{p-1}, \Delta_1^{p-1}, \breve{\varphi}$, respectively. Consequently, there exists a homotopy $\breve{\varphi}_t : \Delta_1^{p-1} \to S_1^{n-1}$ satisfying all the requirements of the lemma. Denote by $\Phi_t : \operatorname{St}_{\Delta^p} a \to \operatorname{St}_{S^n} \bar{a}$ the homotopy which on $\operatorname{St}_{\Delta^p} a \setminus a$ is equal to: $\varphi\pi_{3t}$ for $0 \le t \le {}^1/_3$; $\bar{\pi}_{3t-1}\varphi\pi$ for ${}^1/_3 \le t \le {}^2/_3$; $\breve{\varphi}_{3t-2}\pi$ for ${}^2/_3 \le t \le 1$, and for all t, $\Phi_t a = \bar{a}$. We now take for $\varphi'_t/\operatorname{St}_{\Delta^p} a$ the homotopy which, for a given

t, sends a point x with parameter α to the point lying on the same ray of the star $\bar{a}$ in S^n as $\Phi_t(x)$, but which divides in the ratio $t : (1-t)$ the distance between the points of this ray with parameters $\alpha(\varphi(x))$ and $\alpha(x)$. Then $\varphi'_0 = \varphi / \operatorname{St}_{\Delta^p} a$, and φ'_1 maps each Δ^{p-1}_α piecewise linearly and homeomorphically onto the corresponding S^{n-1}_α, and the ray of the star a to the ray of the star $\bar{a}$; hence φ'_1 maps the star of a in Δ^p piecewise linearly and homeomorphically onto the star of $\bar{a}$ in S^n. By the homotopy extension theorem, we can extend φ'_t in the proper way to Δ^p, after which it may be necessary to apply once more a sweeping-out operation, in order to satisfy the requirement $\varphi'_1(\operatorname{St}_{\Delta^p} a) \cap \varphi'_1(\Delta^p \setminus \operatorname{St}_{\Delta^p} a) = \Lambda$ (which is required for the possibility of a subsequent application of this construction). What follows depends on the proof of the following proposition:

Proposition C. *Let there be given a continuous mapping $\varphi_0 : \Delta^p \to S^n$, piecewise linear and homeomorphic on $\operatorname{St}_{\Delta^p} \dot{\Delta}^p$, with $\varphi_0 \dot{\Delta}^p$ lying in a p-dimensional polyhedron $P \subset S^n$, $p < {}^2/_3 n - 1$, $\varphi_0(\operatorname{Int} \Delta^p) \cap P = \Lambda$.*

Then there exists a piecewise linear embedding $\varphi_1 : \Delta^p \to S^n$, homotopic to φ_0 under a homotopy constant on the boundary Δ^p and for all t mapping $\operatorname{Int} \Delta^p$ into $S \setminus P$.

The proof of this proposition can easily be carried out by Zeeman' s method (see ([3]), p. 66).

3. We return to assertion A. Since the embedding q given to us is locally flat on $\operatorname{Int} \Delta^p$, we can extend it to a neighborhood of $\operatorname{Int} \Delta^p$ in E^n, and, consequently, we may assume that we are given a mapping q of the pair (Δ^n, Δ^p), where Δ^r is the cube $|x_i| \leqslant 1,\ 1 \leqslant i \leqslant r$, with q piecewise linear and homeomorphically mapping Δ^p onto $S^{p-1} \subset P$, and $\Delta^n \setminus \Delta^p$ outside P. According to the lemma, we may also assume that we are given a piecewise linear homeomorphic mapping $q_1 : \Delta^p \to E^n$, homotopic to q/Δ^p under a homotopy fixed on $\dot{\Delta}^p$, and for all t mapping $\operatorname{Int} \Delta^p$ into $E^n \setminus P$. This homotopy, just as in the proof of the lemma in ([1]), can be used to construct an ε-homeomorphism $\check{h} : E^n \to E^n$, identical on P and outside the ε-neighborhood of $q\Delta^p$, and such that the image of a neighborhood of Δ^p in Δ^n under the homeomorphism $\check{h}q$ contains the image of a neighborhood of $\dot{\Delta}^p$ in Δ^p under the homeomorphism q_1. Considering the image of this neighborhood of Δ^p in Δ^p under the homeomorphism $\bar{q} = q^{-1}\check{h}q_1$, we see that it remains only to establish the validity of the following proposition:

Proposition D. Let $\bar{q} : U(\Delta^p) \to \Delta^n$ be a homeomorphic mapping of a neighborhood of $\dot{\Delta}^p$ in Δ^p, identical on $\dot{\Delta}^p$, carrying points of $\operatorname{Int} \Delta^p$ into $\operatorname{Int} \Delta^n$, and locally flat on $\operatorname{Int} \Delta^p$. Then for every $\varepsilon > 0$ there exists an ε-homeomorphism $e : \Delta^n \to \Delta^n$, identical on the boundary of Δ^n and outside the ε-neighborhood of Δ^p in Δ^n, and such that on some neighborhood of $\dot{\Delta}^p$ in Δ^n e coincides with $\bar{q}$.

The proof of this proposition is carried out by the same method as the proof of the main theorem in ([1]) (see ([10])).

Thus, we may assume that the embedding $q : Q \to E^n$ given to us is piecewise linear on some neighborhood of $|T^{p-1}|$ in $|T^p|$. Moreover, we may assume that the diameters of the images of the p-simplices are smaller than a prescribed $\varepsilon > 0$. For each p-simplex $\Delta_i \in T$, we can, as before, construct a homeomorphic mapping q_i of the pair Δ^n, Δ^p into E^n so that q_i on Δ^p coincides with q/Δ_i (i.e., so that $q = qr$ for some piecewise linear homeomorphic mapping $r : \Delta^p \to \Delta_i$) and so that the $\bar{q}_i(\Delta^n \setminus \Delta^p)$ for different i do not intersect one another and do not intersect $q(|T^p| \setminus \Delta_i)$. Let, in addition, the diameter of each $\bar{q}_i(\Delta^n)$ be less than ε. From the second part of the proof of the lemma it is seen that there exists a piecewise linear embedding $q_i' : \Delta^p \to E^n$ such that q_i' coincides with $\bar{q}$ on some neighborhood of $\dot{\Delta}^p$ in Δ^p and $q_i'(\operatorname{Int}\Delta^p) \subset \bar{q}_i(\operatorname{Int}\Delta^n)$. Considering $\bar{q}_i(\operatorname{Int}\Delta^n)$ as a Euclidean space, we see that it is enough to apply, in a slightly stronger form, Theorem 3 of ([1]), according to which two locally flat embeddings of a simplex, coinciding on the boundary, are isotopic under an isotopy identical on the boundary and outside a bounded region.

4. The following two theorems complement Theorem 1; their proofs are carried out within the same technique as in the proof outlined above.

Theorem 2. For every embedding q of a compact polyhedron Q in E^n, of dimension less than ${}^2/_3 n - 1$, which is locally flat on each simplex of some triangulation, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every other embedding of the same kind, δ-close to q, is ε-isotopic to it under an isotopy identical outside the ε-neighborhood of $q(Q)$.

Theorem 3. For every piecewise linear embedding q of a compact polyhedron Q in E^n of dimension less than ${}^2/_3 n - 1$, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every piecewise linear embedding $q' : Q \to E^n$ δ-close to q, one can construct a piecewise linear ε-isotopy $g_t : Q \to E^n$, identical outside the ε-neighborhood of Q, such that $q = g_1 q'$.

An isotopy is called an ε-isotopy if the diameters of all sets $g_t(x)$, $x \in E^n$, are less than ε. An isotopy is called piecewise linear if g_t is piecewise linear for each t and $g_t(x)$ is piecewise linear for each x in t.

Corollary 1. If Q is a polyhedron of dimension less than ${}^2/_3 n - 1$, and if $g_t : Q \to E^n$ is a homotopy that is an embedding for each t, locally flat on the open simplexes of some triangulation of Q, then there exists an isotopy $h_t : E^n \to E^n$ such that $g_1 = h_1 g_0$.

Corollary 2. If Q is a polyhedron of dimension less than ${}^2/_3 n - 1$, and if $g_t : Q \to E^n$ is a homotopy that is a piecewise-linear embedding for each t, then there exists a piecewise-linear isotopy $h_t : E^n \to E^n$ such that $g_1 = h_1 g_0$.

5. Remark 1. Gluck ([6]) and Greathouse ([7]), relying substantially on the technique of Homma ([8]), obtained results analogous to those formulated above, but for $k \leq n/2 - 1$. Their restriction on the embedding of a polyhedron is formally stronger than the one adopted here; however, from Theorem 1 it follows that both conditions turn out to be equivalent for $k \leq n/2 - 1$.

Remark 2. The results of this note can be extended to embeddings of an infinite polyhedron, and also to embeddings of a polyhedron in a combinatorial manifold, analogously to what was done by Gluck and Greathouse. However, in contrast to ([6]), for $k > n/2 - 1$ it remains unproved that if the condition of local flatness on open simplexes is satisfied in one triangulation of the polyhedron, then it will also be satisfied in every other one. This is true, however, for subdivisions of the given triangulation.

Remark 3. If $M^k \subset E^n$ is a combinatorial manifold, then it is obvious that if it is locally flatly embedded in E^n, then in each of its combinatorial triangulations every simplex will be locally flatly embedded. Conversely, if this condition is satisfied for some combinatorial triangulation of M^k, then, by Theorem 1 and Zeeman' s result ([9]), M^k is embedded in E^n locally flatly. The combinatoriality condition is essential here, because it is not known whether an open simplex of a triangulation of a manifold is always embedded in the manifold locally flatly.

Remark 4. The results of this note make it possible to remove the restriction $k \leq n - 2$ in the work of Bing and Kister on embedding in a hyperplane ([9]) (although the resulting isotopies will not be so economical).

Remark 5. In addition to the theorem on the union of locally flat cells formulated in ([2]), we give here one more result:

For $n \geq 4$ and any $p \leq n$, no p-dimensional cell in E^n can have an isolated point of local nonflatness on its boundary.

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Note: Figure translations are in progress. See original paper for figures.