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Abstract

Full Text

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ALMOST METRIZABLE TOPOLOGICAL GROUPS

(Presented by Academician P. S. Aleksandrov on 20 X 1964)

The almost metrizable groups introduced in this note are metrizable with accuracy up to arbitrarily small bicomcompact subgroups. This class of groups is very broad and contains both metrizable and locally bicomcompact groups, and even groups that are complete in the sense of Čech. In relation to metrizable groups, almost metrizable groups behave approximately as locally bicomcompact groups do in relation to Lie groups.

I. Definition of almost metrizable groups

A subset A has countable character in a space X^* if A possesses such a countable system of neighborhoods O_n , $n = 1, 2, \dots$, that every neighborhood of this set contains one of the neighborhoods O_n .

Basic definition. A topological group G will be called **almost metrizable** if in G there is a bicomcompact set of countable character.

Metrizable groups, as is known ⁽¹⁾, coincide with groups satisfying the first axiom of countability. But it is not difficult to see that the fulfillment of the first axiom of countability for the space of a group is equivalent to the existence in it of at least one point of countable character. Thus, the class of groups introduced is in some sense parallel to the class of metrizable groups and contains it.

Since in every open subset of a bicomcompactum X there are sub-bicomcompacta of type G_δ in X , i.e. having countable character in X , locally bicomcompact groups are almost metrizable. Moreover, by virtue of their homogeneity, almost metrizable groups coincide with groups whose spaces have point-countable type in the sense of Arhangel'skii ⁽²⁾, and hence groups complete in the sense of Čech^{**} (the spaces of which have point-countable type ⁽²⁾) are also almost metrizable.

II. Structure of almost metrizable groups

Lemma 1. Let a bicomcompactum Φ have countable character in a space X . Then every bicomcompactum $F \subseteq \Phi$ of type G_δ in Φ also has countable character in X .

Lemma 2. If a bicomcompactum Φ contains the identity of a group G and has countable character in G , then Φ contains a bicomcompact subgroup H^{***} of countable character in G .

The proof is based on the preceding lemma and on the fact that every neighborhood of the identity of the group G contains a subgroup of type G_δ in G .

Lemma 3. If a subgroup H of a group G is bicomact, then the natural mapping

$f : G \rightarrow G/H$ is perfect**** (and open).

Lemma 4. If a subgroup H of a group G has countable character in G , then the quotient space G/H satisfies the first axiom of countability.

* All spaces considered are completely regular.

** That is, groups whose spaces are complete in the sense of Čech.

*** In the note only closed subgroups are considered.

**** A mapping is perfect if it is closed (images of closed sets are closed) and bicomact (preimages of points are bicomact).

Proposition 1. For the quotient space G/H of the group G by the subgroup H , the requirements: 1) to be metrizable, 2) to have the first axiom of countability –are equivalent.*

Theorem 1. The following assertions are equivalent:

- (a) G is an almost metrizable group;
- (b) G has a bicomact subgroup H_0 of countable character, the quotient space G/H_0 by which is metrizable, and the natural mapping $f : G \rightarrow G/H_0$ is open and perfect;
- (c) G contains in every neighborhood of its identity bicomact subgroups of countable character H_α (which are normal divisors in H_0), the quotient spaces G/H_α by which are metrizable, and the natural mappings $f_\alpha : G \rightarrow G/H_\alpha$ are open and perfect;
- (d) G is the limit of a spectrum $S = \{X_\alpha, \delta_\alpha^\beta\}$, $\alpha \in \mathfrak{A}$, of metrizable quotient spaces $X_\alpha = G/H_\alpha$, and the naturally arising projections $\delta : X_\beta \rightarrow X_\alpha$ for $H_\beta \subseteq H_\alpha$ are open and perfect;
- (e) the space of the group G is feathered in the sense of Arhangel'skiĭ ⁽²⁾.

The proof of the equivalence of items (a) and (b) follows from the preceding assertions. Item (c) follows from item (b), because a bicomact subgroup contains arbitrarily small normal divisors, the quotient spaces by which are Lie groups. Item (d) is derived from item (c) in the usual way.

The equivalence of items (a) and (e) follows from the fact that: 1) a feathered space has a point-countable type ⁽²⁾; 2) an almost metrizable group, by item (b), has a perfect mapping onto a metric space, i.e. its space is feathered ⁽²⁾.

As A. A. Markov showed ⁽³⁾, the space of a group need not be normal. For almost metrizable groups this is not so:

Corollary 1. Almost metrizable groups are paracompact (and hence also normal).

Corollary 2. Groups that are complete in the sense of Čech are paracompact.

We recall that the requirement of local bicomactness of a group, which is stronger than completeness in the sense of Čech, already entails its strong paracompactness ^(4, 5).

Corollary 3. The following assertions are equivalent:

- a) the group G is complete in the sense of Čech;
- b) G contains arbitrarily small bicomact subgroups H_α , the quotient spaces X_α by which are metrizable and complete in the sense of Čech, and the natural mappings $f_\alpha : G \rightarrow X_\alpha$ are open and perfect.

Proposition 2. An almost metrizable group is finally compact if and only if it has a countable everywhere dense system of bicomacts.

III. Dimension of almost metrizable groups.

As follows from Katětov's theorem ⁽⁶⁾, for metrizable groups the dimensions \dim and Ind coincide; for locally bicomact groups, the dimension ind also coincides with the named dimensions ⁽⁷⁾. These results are consequences of a more general fact:

Theorem 2. For an almost metrizable group G one always has

$$\dim G = \text{Ind } G.$$

For a strongly paracompact (even for a fully paracompact ⁽⁸⁾) group G one always has

$$\dim G = \text{ind } G = \text{Ind } G.$$

Corollary 4. For a group G complete in the sense of Čech one always has

$$\dim G = \text{Ind } G.$$

* This assertion was also proved by A. Arhangel'skiĭ.

Since locally bicomact groups are strongly paracompact, Theorem 2 also implies the coincidence of the various definitions of dimension for locally bicomact groups.

The proof of Theorem 2 is based on the following theorem, which refines Theorem 1.

Theorem 3. The following assertions are equivalent:

(a') the nearly metrizable group G contains a finite-dimensional bicomactum in the sense of \dim ;

(b', c') in items (b) and (c) of Theorem 1 the groups H_0 and H_α may be taken to be zero-dimensional;

(f') in item (f) of Theorem 1 the projections δ_α^β may be taken to be strictly finite-to-one, i.e. locally topological.

Corollary 5. If for a nearly metrizable group G we have $\text{ind } G < \infty$, then G possesses a zero-dimensional perfect mapping onto a metric space.

One can prove even a stronger assertion than in Theorem 2:

Theorem 4. If the group G is nearly metrizable and $\dim G = k$, then G is a $(k+1)$ -fold and closed image of a strongly paracompact zero-dimensional space, i.e. in this case G is perfectly k -dimensional in the sense of Ponomarev ⁽⁹⁾.

Corollary 6. If a nearly metrizable group G contains a finite-dimensional in the sense of \dim bicomactum of countable character (for example, if $\text{ind } G < \infty$), then:

(a) for every closed set $F \subseteq G$ one always has

$$\dim F = \text{Ind } F;$$

(b) for every strongly (fully) paracompact set $A \subseteq G$ one always has

$$\dim A = \text{ind } A = \text{Ind } A * .$$

Moreover, if $\dim F = k$ (respectively $\dim A = k$), then F (respectively A) is a $(k+1)$ -fold closed image of a zero-dimensional strongly paracompact space.

IV. Metrization of Nearly Metrizable Groups.

Lemma 5. A group possessing a compactum of countable character is metrizable.

From the fact that a bicomact group of weight τ contains a bicomactum D^τ , it follows that

Theorem 5. A nearly metrizable group is metrizable if and only if its space is hereditarily normal (for example, perfectly normal).

Corollary 7. A Čech-complete hereditarily normal group is metrizable.

Let us also note that all locally connected bicomacta contained in a nearly metrizable group G are metrizable, if G contains a finite-dimensional in the sense of \dim bicomactum of countable character (for example, if $\text{ind } G < \infty$).

V. Factor Spaces of Nearly Metrizable Groups.

The following theorem concerns the structure of factor spaces of nearly metrizable groups, which is analogous to the structure of the groups themselves.

Theorem 6. If G is a nearly metrizable group and H is its subgroup, then the factor space G/H possesses a perfect open mapping π onto a metric space.

If G contains a finite-dimensional bicomactum of countable character (for example, if $\text{ind } G < \infty$), then the mapping π may be taken to be zero-dimensional.

* These equalities hold even for such a set $A \subseteq G$, for every closed subset F of which of type G_δ in A the relation $\dim F \leq \text{ind } F$ is satisfied.

Corollary 8. The quotient space of an almost metrizable group is paracompact (i.e., also normal).

Theorem 7. The quotient space of an almost metrizable group is metrizable if and only if it is hereditarily normal.

Theorem 8. Let G be an almost metrizable group, and let H be its subgroup. Then for the quotient space G/H the relation

$$\dim G/H = \text{Ind } G/H$$

holds provided one of the following two conditions is satisfied:

- 1) $\text{ind } H < \infty$;
- 2) G contains a finite-dimensional, in the sense of \dim , bicomact set of countable character.

In particular, the indicated equality holds when $\text{ind } G < \infty$.

For subspaces of quotient spaces G/H of almost metrizable groups G (if the groups G contain finite-dimensional, in the sense of \dim , bicomacta of countable character), a statement analogous to Corollary 6 is valid.

VI. Completeness of metrizable groups. As was shown by Kakutani in ⁽¹⁾, every metrizable group has a left-invariant (right-invariant) metric. It is known that, for metric spaces, the existence of a complete metric is equivalent to completeness in the sense of Čech (the Alexandroff-Hausdorff theorem ^(11, 12)). It is natural to ask: for metrizable groups, is the requirement of completeness in the sense of Čech (i.e., the requirement that a complete metric exist on them) equivalent to the requirement that a complete left-invariant (right-invariant) metric exist on them? The answer to this question turns out to be negative.

Theorem 9. A metrizable group has a complete left (right) invariant metric if and only if it is complete in the sense of Weil.*

Theorem 10. A metrizable group is complete in the sense of Raikov [i.e., absolutely closed] if and only if it is complete in the sense of Čech [i.e., has a complete metric].

In ⁽¹⁰⁾ an example is given of a group that is completable in the sense of Raikov but not completable in the sense of Weil, and moreover this group satisfies the first axiom of countability. The Raikov completion of this group: 1) will also satisfy the first axiom of countability, i.e., will be metrizable; 2) will be complete in the sense of Raikov, i.e., by Theorem 10, will have a complete metric; 3) will not be complete in the sense of Weil, i.e., by Theorem 9, will not have a complete left (right) invariant metric.

Some questions:

1. For almost metrizable groups, is completeness in the sense of Raikov equivalent to completeness in the sense of Čech?
2. Is it always true that $\dim X = \text{Ind } X$ for the quotient space $X = G/H$ of an almost metrizable group G ? Does the relation $\dim X < \infty$ imply the existence of a zero-dimensional perfect mapping of X onto a metric space?

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* The definition of completeness in the sense of Weil and in the sense of Raikov (see below) can be found in ⁽¹⁰⁾.

Note: Figure translations are in progress. See original paper for figures.

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