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Academician of the Academy of Sciences of the Moldavian SSR V. A. ANDRUNAKIEVICH, Yu. M. RYABUKHIN

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Abstract

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MATHEMATICS

Academician of the Academy of Sciences of the Moldavian SSR V. A. ANDRUNAKIEVICH, Yu. M. RYABUKHIN

LINKED RINGS

In the present note the concept is introduced of an element linked with zero, which is a special case of the concept of a zero divisor; rings in which all elements are linked with zero are studied.

Let K be an arbitrary (associative) ring. If A is a (two-sided) ideal of the ring K contained in the ideal B, then the left annihilator of the ideal A in the ring B is the set $l_B(A) = \{x \mid x \in B,\ xA = 0\}$. If $B = K$, then instead of $l_K(A)$ we shall write $l(A)$. It is clear that $l_B(A)$ is an ideal in K and $l_B(A) = B \cap l(A)$. Analogously one defines the right annihilator $r_B(A)$ of the ideal A in the ring B. As usual, by $(a)_K$, or, more briefly, by (a), we denote the principal ideal in K generated by the element $a \in K$. Instead of $l_B((a))$ and $l((a))$ we shall write respectively $l_B(a)$ and $l(a)$.

We shall say that a nonzero element $a \in K$ is linked with zero (on the right) if $l_{(a)}(a) \neq 0$. The zero element is considered to be linked with zero. We shall call the ring K linked (on the right) if all its elements are linked (on the right) with zero. Analogously one defines elements linked with zero on the left, and rings linked on the left. There exist rings that are linked on the right but not on the left.

It is clear that nilpotent rings are linked. Every commutative nil-ring is, evidently, linked. Moreover, if a ring is the sum of all its nilpotent ideals, then it is also linked on the right. We note that if a ring K is linked on the right, then every nonzero ideal of K contains a nonzero nilpotent ideal.

Proposition 1. *Every linked ring is representable in the form of a subdirect sum of subdirectly irreducible rings with nilpotent heart.*

Recall that a ring K is called radical in the sense of Baer or, more briefly, a Baer ring, if every nonzero homomorphic image of the ring K has a nonzero nilpotent ideal. Baer rings are nil-rings, and every ideal and every homomorphic image of a Baer ring are Baer rings. It is clear that every nonzero ideal of a Baer ring contains a nonzero nilpotent ideal.

A natural question arises: what relation exists between Baer rings and linked rings?

From what was said above it is clear that a commutative Baer ring is linked on the right.

Proposition 2. *A subdirectly irreducible Baer ring is a linked ring.*

The following example shows that there exist Baer rings which are not linked.

Example 1. Let $K = Z[x_1, x_2, ...]$ be the ring of polynomials in a countable number of variables over the ring of integers, and let $Q = (x_1^2, x_2^3, ...)$ be the ideal in K generated by the elements x_i^{i+1}, $i = 1, 2, ...$. We note that Q consists of all polynomials without constant term having the property that in each of their terms at least one of the x_i occurs to exponent at least $i + 1$. Since for every i, $(x_i)^{i+1} = (x_i^{i+1}) \subseteq Q$, all elements of the factor ring $\overline{K} = K/Q$ that are images of poly-

members without free members generate nilpotent ideals and therefore are contained in the Baer radical $L(\overline{K})$ of the ring $\overline{K}$. Let U be the subring of the ring K_2 of all second-order matrices over $\overline{K}$, consisting of all matrices of the form

$$\begin{pmatrix} \overline{f}_{11} & \overline{g} \\ \overline{f}_{21} & \overline{f}_{22} \end{pmatrix},$$

where $\overline{f}_{ij} \in L(\overline{K})$ and $\overline{g} \in \overline{K}$. It is easy to verify that U is a nil ring and satisfies a polynomial identity (see (1), pp. 335 and 336), i.e., U is a Baer ring. Meanwhile it is not hard to check that if

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then $l_{(a)}(a) = 0$, although $a \in U$ and $a \neq 0$. Consequently, the ring U is not a right linked ring, although it is a Baer ring.

As usual, a commutative ring with the maximum condition for ideals will be called a Noetherian ring.

Theorem 1. *A Noetherian linked ring K is nilpotent, and hence is a Baer ring.*

Proof. It suffices to prove that the ring K is a nil ring, since Noetherian nil rings, as is known, are nilpotent. Suppose that K is not a nil ring. Then there exists an element b in K such that $b^k \neq 0$ for all natural k. Consider the increasing chain of ideals

$$l(b) \subseteq l(b^3) \subseteq ... \subseteq l(b^{2^n - 1}) \subseteq l(b^{2^{n+1} - 1}) \subseteq ... \qquad (1)$$

By the choice of the element b, the principal ideal $B_n = (b^{2^n}) \neq 0$ for all n. In view of the linkedness of the ring K, $l_{B_n}(B_n) \neq 0$. Let $x_n \in l_{B_n}(B_n)$ and $x_n \neq 0$. Then there exists an element $z_n \in K$ such that

$$x_n = mb^{2^n} + z_n b^{2^n} = (mb + z_n b)b^{2^n - 1}. \tag{2}$$

Putting $y_n = mb + z_n b$, we obtain $x_n = y_n b^{2^n} \neq 0$, and therefore $y_n \notin l(b^{2^n-1})$. Moreover,

$$y_n b^{2^{n+1}-1} = y_n b^{2^n-1} b^{2^n} = x_n b^{2^n} = 0,$$

and therefore $y_n \in l(b^{2^{n+1}-1})$. Consequently, the infinite chain (1) of ideals is strictly increasing, which contradicts the Noetherian property of the ring K. The contradiction obtained shows that K is a nil ring.

Analogously one proves that:

A commutative linked ring with the minimum condition for ideals is nilpotent, and hence is a Baer ring.

In the last assertion and in Theorem 1 the requirement that the ring be commutative cannot be dropped, since there exist finite right linked rings having idempotent elements and therefore not being Baer rings. There also exist commutative linked rings that are not Baer rings (^{2}).

We shall call a ring K uniformly linked (on the right) if for every nonzero ideal A of K, $l_A(A) \neq 0$. Obviously, every uniformly linked ring is linked. The converse assertion is not true even in the commutative case. Nevertheless, under the minimum or maximum condition for ideals, every linked ring is uniformly linked. To prove these assertions we shall need some notions and lemmas.

Let A be an ideal of the ring K, and B an ideal in A. We shall say that B is an r-ideal in A if B is maximal among the ideals of the ring K having the form $r_A(x)$, where $x \in A$, $x \neq 0$, (x) is the principal ideal in K generated by the element x.

Lemma 1. *Let A be an ideal in K, and let B be an r-ideal in A. Then B is a prime ideal in the ring A.*

Lemma 2. *If the ring K satisfies the maximum condition for ideals, then any ideal A has only finitely many r-ideals.*

Lemma 3. If an ideal A is contained in a finite union $\bigcup_{i=1}^{m} P_i$ of prime ideals, then it is contained in one of them.

Theorem 2. Every connected ring K satisfying the maximality condition for ideals is uniformly connected.

Proof. Since the ring K is connected, for any element a of a nonzero ideal A there exists an element $x \in A$, $x \neq 0$, such that $x(a) = 0$. But then $a \in r_A(x)$.

By the maximality condition for ideals, every right annihilator $r_A(x)$, where $x \in A$, $x \neq 0$, is contained in some r-ideal. Consequently,

$$A = \bigcup \{r_A(x_i) \mid r_A(x_i) \text{ are } r\text{-ideals of the ring } A\}.$$

By Lemma 1, all r-ideals $r_A(x_i)$ are prime ideals of the ring A, and by Lemma 2 there are only finitely many such ideals. Finally, by Lemma 3, $A \subseteq r_A(x_i)$ for some r-ideal $r_A(x_i)$. But this means that $x_i A = 0$. Since, by the definition of r-ideals, $x_i \in A$ and $x_i \neq 0$, we have $l_A(A) \neq 0$. By the arbitrariness of the choice of the ideal A, the ring K is uniformly connected.

Lemma 4. Let A be an ideal of the ring K. If the ring K satisfies the minimality condition for ideals, then every right annihilator $r_A(x)$, where $x \in A$, $x \neq 0$, is contained in some r-ideal, and A has only a finite number of r-ideals.

Using Lemma 4 and repeating almost verbatim the proof of Theorem 2, we obtain the following assertion:

Theorem 3. A connected ring K with the minimality condition for ideals is uniformly connected.

We shall call an ideal A of the ring K connected (on the right) with zero if each of its elements is connected on the right with zero. We shall say that an ideal A is strictly connected (on the right) with zero if, for every connected (on the right) ideal B of the ring K, the sum $A + B$ is a connected ideal. Obviously, a strictly connected ideal is a connected ideal, since the zero ideal is connected with zero.

Proposition 3. In any ring K, the sum

$$A = \sum_{\alpha} A^{\alpha}$$

of any set of strictly connected ideals A^{α} is a strictly connected ideal. In particular, the sum $M_r(K)$ of all strictly connected ideals is the largest strictly connected ideal of the ring K.

Let us note that, by the Kuratowski–Zorn lemma, every connected ideal is contained in some maximal connected ideal.

Proposition 4. The largest strictly connected ideal $M_r(K)$ of the ring K coincides with the intersection of all maximal connected ideals of the ring K.

Generally speaking, it is not true for every ring K that

$$M_r(K/M_r(K)) = 0.$$

Indeed, consider the ring U constructed in Example 1. From the construction of the ring U it follows easily that the sum of all nilpotent ideals of the ring U coincides with the set

$$V = \left\{ \begin{pmatrix} \bar{f}_{11} & \bar{f}_{12} \\ \bar{f}_{21} & \bar{f}_{22} \end{pmatrix} \middle| \, f_{ij} \text{ are images of polynomials without constant terms} \right\}.$$

Let us now note that in the ring U only the elements of V are connected on the right with zero. Since V is an ideal of the ring U, V is the largest connected ideal in U, and therefore $V = M_r(U)$. Observing that for any g_1, g_2 in $\bar{K}$ (see Example 1) we have

$$\begin{pmatrix} 0 & \bar{g}_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{g}_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain that the factor ring U/V is a nilpotent ring. Putting $K = U$, we obtain

$$M_r(K/M_r(K)) = M_r(U/V) = U/V \neq 0.$$

Recall that an ideal Q of the ring K is called semiprimary if the factor ring K/Q has no nonzero nilpotent ideals.

Theorem 4. If a ring K satisfies the maximality (or minimality) condition for ideals, then $M_r(K)$ is a semiprimary

ideal and coincides with the intersection: 1) of all prime ideals containing $M_r(K)$; 2) of all prime ideals containing at least one maximal connected ideal of the ring K.

Corollary. If a ring K satisfies the maximality (or minimality) condition for ideals, then $M_r(K/M_r(K)) = 0$, and the rings K for which $M_r(K) = 0$ are precisely the rings without nonzero nilpotent ideals.

Analogous results hold for commutative rings.

Theorem 5. Every ideal of a connected ring with the maximality (or minimality) condition for ideals is also a connected ring.

An analogous result is also valid for commutative rings.

In conclusion we shall construct an example showing that there exist a connected (right) ring A and an ideal B in A such that B is not a connected (right) ring, although it is an ideal connected (on the right) with zero.

Example 2. Let P be the semigroup over the alphabet Ξ, $\Xi = \{x_1, x_2 \ldots, \ldots, x_n, \ldots;\ y;\ z\}$, with zero 0, defined by the following defining relations: a) if $i \geq j$, then $x_i \cdot x_j = 0$; b) $y \cdot x_i = x_{i+1}$; c) $z \cdot x_i = z \cdot y = z \cdot z = y \cdot z = x_{i_1} x_{i_2} x_{i_3} z = 0$.

Every element of the semigroup P has one and only one of the following forms: 1) $x_{i_1} x_{i_2} \ldots x_{i_n}$, where $i_1 < i_2 < \ldots < i_n$, $n \geq 1$; 2) $x_{i_1} x_{i_2} \ldots x_{i_n} y^k$, where $i_1 < i_2 < \ldots < i_n$, $n \geq 1$, $k \geq 1$; 3) y^k, $k \geq 1$; 4) $x_{i_1} x_{i_2} z$, where $i_1 < i_2$; 5) $x_i z$; 6) z; 7) 0.

Let K be the algebra over the field Φ with basis $P \setminus \{0\}$ (see (3), pp. 264-265). The elements of the ring K are written in the form

$$a = \sum \alpha_i^{(1)} p_i^{(1)} + \sum \alpha_j^{(2)} p_j^{(2)} + \sum \alpha_k^{(3)} p_k^{(3)} + \sum \alpha_n^{(4)} p_n^{(4)} +$$

$$+\sum \alpha_m^{(5)} p_m^{(5)} + \alpha^{(6)} z,$$

where $\alpha_s^{(t)} \in \Phi$, $p_s^{(t)} \in P$, $p_s^{(t)}$ has form t, $t = 1, 2, 3, 4, 5$. The set A of elements a having the form

$$a = \sum \alpha_i^{(1)} p_i^{(1)} + \sum \alpha_j^{(2)} p_j^{(2)} + \sum \alpha_k^{(3)} p_k^{(3)} + \sum \alpha_n^{(4)} p_n^{(4)} + \sum \alpha_m^{(5)} p_m^{(5)},$$

is an ideal of the ring K. The set B of elements b of the form

$$b = \sum \alpha_i^{(1)} p_i^{(1)} + \sum \alpha_j^{(2)} p_j^{(2)} + \sum \alpha_n^{(4)} p_n^{(4)}$$

is an ideal of the ring A. It can be shown that the rings K and A are connected rings, but the ring B is not connected, although it is an ideal of the connected ring A.

Institute of Mathematics with Computing Center
Academy of Sciences of the MSSR

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Note: Figure translations are in progress. See original paper for figures.

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