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Abstract

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ON THE INDICATOR OF FUNCTIONS OF NON-INTEGER ORDER, ANALYTIC AND OF COMPLETELY REGULAR GROWTH IN A HALF-PLANE

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In the works of B. Ya. Levin (^{1}), Chs. I, III; (^{2}) and A. Pfluger (^{3}), the equivalence was established between the existence of an angular density for the set of zeros of an entire function of non-integer order and the complete regularity of its growth (for definitions of these concepts see (^{1}), pp. 118, 182). In this connection a formula was found expressing the indicator of such a function in terms of the angular density of the set of its zeros, and conversely.

In the present article, instead of entire functions, functions regular in a half-plane are considered. For them, in a special way, the concept of order and analogues of the concept of angular density are introduced, characterizing the complete regularity of growth of the function; moreover, a formula for the indicator of these functions is derived.

Definition 1. Let the function $f(z)$ be regular in $\alpha < \arg z < \beta$ and, for some $\mu > 0$, satisfy the asymptotic estimate

$$\sup_{|z|\le r,\ \alpha<\arg z<\beta} |f(z)| < \exp(r^\mu); \tag{1}$$

denote by $\{\mu\}$ the set of all such μ. Let, furthermore, $\{\nu\}$ be the set of all such values $\nu > 0$ for which

$$\overline{\lim_{r\to\infty}} \ln|f(re^{i\theta})|/r^\nu \equiv 0 \quad (\alpha < \theta < \beta)^*. \tag{2}$$

Then the number $\rho = \max(\inf\{\mu\}, \inf\{\nu\})$ shall be called the **order of the function** $f(z)$ **inside the angle** $\alpha < \arg z < \beta$, and the function

$$h_f(\theta) = \overline{\lim_{r\to\infty}} \ln|f(re^{i\theta})|/r^\rho \tag{3}$$

shall be called its **indicator**.

If (1) is not fulfilled for any $\mu > 0$, then $f(z)$ is called a **function of infinite order**.

A definition equivalent to this one is given in (^{4}), p. 209. For an entire function (with $0 \le \arg z \le 2\pi$) it coincides with the commonly accepted one:

$$\rho = \overline{\lim_{r\to\infty}} \ln\ln M_f(r)/\ln r, \qquad \text{where } M_f(r) = \max_{|z|=r} |f(z)|.$$

Let $f(z)$ be regular and of finite order in $\operatorname{Im} z > 0$. Then from (1) it follows ((^{5}), pp. 66, 83) that it has, almost everywhere for $-\infty < t < \infty$, finite angular boundary values, and the limiting function $\ln|f(t)|$ is summable on every finite interval $-r \le t \le r$.

A function $f(z)$ of order $\rho > 0$ inside the angle $\alpha < \arg z < \beta$ shall be called a **function of finite type** if, asymptotically,

$$\sup_{|z|\le r,\ \alpha<\arg z<\beta} \ln|f(z)| < Kr^\rho, \qquad \text{where } K = \text{const}.$$

Definition 2. A function $f(z)$, regular and of order $\rho > 0$ in the angle $\alpha < \arg z < \beta$, continuous in $\alpha \le \arg z \le \beta$, is called a function of completely regular growth in the closed angle $\alpha \le \arg z \le \beta$, if the function $r^{-\rho}\ln|f(re^{i\theta})|$ tends to $h_f(\theta)$ uniformly in θ, as r tends to $+\infty$, except possibly on a set E, common for all θ, of zero relative measure ((1), pp. 127, 182).

We shall call $f(z)$ a function of completely regular growth in the open angle $\alpha < \arg z < \beta$, if it is of finite type in $\alpha < \arg z < \beta$ and has completely regular growth in every angle $\alpha + \varepsilon \le \arg z \le \beta - \varepsilon$ ($\varepsilon > 0$).

Denote by A_ρ (by $\overline{A}_\rho$) the class of functions, regular in $\operatorname{Im} z > 0$, having order ρ and completely regular growth in the open angle $0 < \arg z < \pi$ (in the closed angle $0 \le \arg z \le \pi$). Obviously, $\overline{A}_\rho \subset A_\rho$.

Theorem 1. *Every function, regular and of order $\rho \ge 0$ in $\operatorname{Im} z > 0$, is representable in the form*

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* It can be proved that (2) is certainly valid for $\nu > \max(\mu, \pi/(\beta - \alpha))$.

$$f(z) = \exp\left[i(a_0 + a_1 z + \dots + a^{q}_{qz})\right] \times$$
$$\exp\left\{\frac{1}{\pi i}\left[\int_{-\infty}^{\infty} \frac{(tz+1)^{q+1}\ln|f(t)|}{(t^2+1)^{q+1}(t-z)}\,dt + \int_{-\infty}^{\infty} \frac{(tz+1)^{q+1}}{(t^2+1)^{q}(t-z)}\,d\varphi(t)\right]\right\} \prod_{|z_n|<1} \frac{z-z_n}{z-\overline{z}_n} \times$$
$$\times \prod_{|z_n|\geq 1} \frac{\left(1-\dfrac{z}{z_n}\right)\exp\left[\dfrac{z}{z_n} + \dfrac{1}{2}\left(\dfrac{z}{z_n}\right)^2 + \dots + \dfrac{1}{q}\left(\dfrac{z}{z_n}\right)^q\right]}{\left(1-\dfrac{z}{\overline{z}_n}\right)\exp\left[\dfrac{z}{\overline{z}_n} + \dfrac{1}{2}\left(\dfrac{z}{\overline{z}_n}\right)^2 + \dots + \dfrac{1}{q}\left(\dfrac{z}{\overline{z}_n}\right)^q\right]}, \tag{4}$$

where $q = [\rho]$*;* a_k *are real constants;* z_n *are the internal, i.e., lying in* $\operatorname{Im} z > 0$*, zeros of* $f(z)$*;* $\varphi(t)$ *is a real nondecreasing function for which almost everywhere* $\varphi'(t) = 0$*, and the integrals* $\displaystyle\int_{-\infty}^{-1} \frac{d\varphi(t)}{t^q}$ *and* $\displaystyle\int_{1}^{\infty} \frac{d\varphi(t)}{t^q}$ *converge.*

The theorem follows from previously known results ((7), p. 189, (6)). We now take an arbitrary function $f(z)$, regular and of order $\rho > 0$ in $\operatorname{Im} z > 0$, and, in the notation of Theorem 1, put

$$\tau(t) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{1}^{t} \frac{\ln|f(x)|}{x}\,dx + \frac{1}{2\pi}\int_{0}^{t} x\,d\varphi(x), & t > 1, \\ \dfrac{1}{2\pi}\displaystyle\int_{t}^{-1} \frac{\ln|f(x)|}{|x|}\,dx + \frac{1}{2\pi}\int_{t}^{0} |x|\,d\varphi(x), & t < -1; \end{cases} \tag{5}$$

$$c(r, \eta_1, \eta_2) = \sum_{\substack{\eta_1 < \arg z_n \leq \eta_2 \\ 1 \leq |z_n| \leq r}} \sin\arg z_n, \qquad r > 1.$$

Next, compose the function

$$a(r, \eta_1, \eta_2) = \begin{cases} c(r, \eta_1, \eta_2), & 0 < \eta_1 < \eta_2 < \pi, \\ c(r, 0, \eta_2) - \tau(r), & 0 = \eta_1 < \eta_2 < \pi, \\ c(r, \eta_1, \pi) - \tau(-r), & 0 < \eta_1 < \eta_2 = \pi, \\ c(r, 0, \pi) - \tau(r) - \tau(-r), & \eta_1 = 0,\ \eta_2 = \pi; \end{cases} \tag{6}$$

$$a(r, \eta_1, \eta_2) = -a(r, \eta_2, \eta_1) \quad \text{for } \eta_1 > \eta_2; \qquad a(r, \eta, \eta) \equiv 0.$$

Definition 3. If, for a function $f(z)$, regular and of order ρ in $\operatorname{Im} z > 0$, for all $\eta_1, \eta_2 \in (0 \leq \eta \leq \pi) \setminus N$, where N is at most countable and does not contain the points $\eta = 0, \eta = \pi$, there exists the finite limit

$$\lim_{r\to\infty} \frac{a(r,\eta_1,\eta_2)}{r^\rho} = \lambda(\eta_1,\eta_2),$$

then we shall say that the set of zeros of the function $f(z)$ has an argument-boundary density in the half-plane $\operatorname{Im} z > 0$.

The concept of argument-boundary density is closely connected with the concept of angular density introduced by B. Ya. Levin (^{1}, p. 118):

$$\lim_{r\to\infty} \frac{1}{r^\rho} \sum_{\substack{\eta_1 < \arg z_n \leq \eta_2 \\ 1 \leq |z_n| \leq r}} 1 = \Delta(\eta_1,\eta_2). \tag{7}$$

We note that the quantity $c(r,0,\pi)$ was considered by R. Nevanlinna (^{6}).

Theorem 2. *In order that a function, regular of noninteger order ρ and finite type in $\operatorname{Im} z > 0$, belong to the class A_ρ, it is necessary and sufficient that the set of its zeros have an argument-boundary density.*

Theorem 3. *If $f(z) \in A_\rho$ and $\rho > 0$ is noninteger, then the indicator of the function $f(z)$ is expressed by the formula*

$$h_f(\theta) = \frac{\pi}{\sin \pi\rho} \int_0^\pi g(\psi,\theta)\, d\lambda(\psi), \qquad 0 < \theta < \pi,$$

where $\lambda(\psi) \equiv \lambda(0,\psi)$ is the argument-boundary density of the zeros of $f(z)$ and

$$g(\psi,\theta) = \begin{cases} \dfrac{1}{\sin\psi}\left[\cos\rho(|\theta-\psi|-\pi) - \cos\rho(\theta+\psi-\pi)\right], & 0 < \psi < \pi, \\ 2\rho \sin\rho(\theta-\pi), & \psi = 0, \\ -2\rho\sin\rho\theta, & \psi = \pi. \end{cases} \tag{8}$$

We now introduce concepts characterizing separately the asymptotic distribution of boundary and interior zeros of a function in $\operatorname{Im} z \geq 0$.

Definition 4. Let the function $f(z)$ be regular and of order $\rho > 0$ in $\operatorname{Im} z > 0$, and let $\tau(r)$ be defined by equality (5). Then the limits

$$\lim_{r\to+\infty} \frac{\tau(r)}{r^\rho} = l_1, \qquad \lim_{r\to+\infty} \frac{\tau(-r)}{r^\rho} = l_2, \tag{9}$$

if they exist and are finite, are called, respectively, the right- and left-sided boundary densities of the set of zeros of the function $f(z)$. If in (9) upper limits are taken, then we shall call them upper boundary densities (l_1^* and l_2^*).

Definition 5. Let in $\operatorname{Im} z > 0$ a set of points $\{z_n\}$, $n = 1, 2, ...$, be given, all of whose limit points lie on the real axis. Then, if for all $\eta_1, \eta_2 \in (0 \leq \eta \leq \pi) \setminus N$, where N is at most countable and does not contain $\eta = 0, \eta = \pi$, there exists a finite limit

$$\lim_{r \to \infty} c(r, \eta_1, \eta_2)/r^\rho = \mu(\eta_1, \eta_2), \tag{10}$$

then we shall say that the set $\{z_n\}$ has, in the domain $\operatorname{Im} z > 0$, an argument density with exponent ρ (or an upper argument density $\mu^*(\eta_1, \eta_2)$, if in (10) an upper limit is taken).

Theorem 4. *If $f(z)$ is a function of the class A_ρ, $\rho > 0$, then: 1) its upper boundary densities are finite, and the upper argument density is bounded; 2) its indicator can be expressed by the formula* $(0 < \theta < \pi)$

$$h_f(\theta) = \frac{\pi}{\sin \pi\rho} \left[\int_0^\pi g(\psi, \theta)\, d\mu^*(\psi) - 2\rho l_1^* \sin \rho(\theta - \pi) + 2\rho l_2^* \sin \rho\theta \right],$$

where $\mu^*(\psi) = \mu^*(0, \psi)$, and $g(\psi, \theta)$ is defined by relation (8).

The argument density of zeros of a function of the class A_ρ exists if and only if there exists for it a two-sided boundary density. For functions of the class $\bar{A}_\rho$, however, there holds

Theorem 5. In order that a function $f(z)$ of noninteger order $\rho > 0$ and finite type in $\operatorname{Im} z > 0$ belong to the class $\bar{A}_\rho$, it is necessary and sufficient that the set of its zeros have an angular density, continuous at $\psi = 0$ and $\psi = \pi$, and a two-sided limiting density satisfying the equalities

$$\lim_{r \to +\infty} \frac{\ln |f(r)|}{r^\rho} = \rho l_1, \qquad \lim_{r \to +\infty} \frac{\ln |f(-r)|}{r^\rho} = \rho l_2.$$

The indicator of the function $f(z)$ is expressed by the formula $(0 \leq \theta \leq \pi)$

$$h_f(\theta) = \frac{\pi}{\sin \pi\rho} \left[\int_0^\pi g(\psi, \theta)\, d\mu(\psi) - 2\rho l_1 \sin \rho(\theta - \pi) + 2\rho l_2 \sin \rho\theta \right].$$

Let us dwell further on the properties of the angular density (7).

Theorem 6. If $f(z) \in A_\rho$, $\rho > 0$, and $\Delta(\psi)$ is the angular density of the set of zeros of $f(z)$, then the Stieltjes integral

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* If $\rho \geq 1$, then 1 is true for any function of finite type in $\operatorname{Im} z < 0$.

$$\int_0^{\pi} \sin\psi \, d\Delta(\psi) = \lim_{\delta,\varepsilon\to+0} \int_{\delta}^{\pi-\varepsilon} \sin\psi \, d\Delta(\psi)$$

(generally speaking, improper) converges.

Theorem 7. If $f(z) \in \bar{A}_\rho$ and ρ is noninteger, then the formula holds

$$h_f(\theta) = \frac{1}{\sin\pi\rho} \left\{ \pi \int_0^{\pi} [\cos\rho(|\theta-\psi|-\pi) - \cos\rho(\theta+\psi-\pi)] \, d\Delta(\psi) \right.$$

$$\left. -h_f(0)\sin\rho(\theta-\pi) + h_f(\pi)\sin\rho\theta \right\} \qquad (0 \le \theta \le \pi).$$

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Note: Figure translations are in progress. See original paper for figures.