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Abstract

Full Text

MATHEMATICS

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ON DIRECT AND INVERSE EMBEDDING THEOREMS FOR WEIGHTED SPACES

(Presented by Academician I. M. Vinogradov, 19 III 1965)

1. Let $n_1, \dots, n_s$ be natural numbers whose sum is equal to n; E^{n_i} $(i = 1, \dots, s)$ are n_i-dimensional Euclidean spaces of points

$$\mathbf{y}^{(i)} = (y_1^{(i)}, \dots, y_{n_i}^{(i)}); \qquad E_s^n = \prod_{i=1}^{s} E^{n_i}$$

are spaces of points

$$\mathbf{y} = (\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(s)}) = (y_1^{(1)}, \dots, y_{n_i}^{(1)}; \dots; y_1^{(s)}, \dots, y_{n_s}^{(s)}) = (y_1, \dots, y_n);$$

if $n_i = 1$, then $\mathbf{y}^{(i)} = y_i$.

Let

$$E^{+n_i} = \{\mathbf{y}^{(i)},\, y_{j_i}^{(i)} > 0 \ (j_i = 1, \dots, n_i)\}, \qquad E_s^{+n} = \prod_{i=1}^{s} E^{+n_i}.$$

For $s = n$, instead of E_n^n and E_n^{+n} we shall write, respectively, E^n and E^{+n}.

2. Let $f(\mathbf{y})$ be an arbitrary smooth function defined in the space E_s^{+n}. Suppose $\mathbf{r} = (\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(s)})$, where $\mathbf{r}^{(i)} = (r_1^{(i)}, \dots, r_{n_i}^{(i)})$, is a vector with positive integer components, $1 \le p \le \infty$; $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a vector whose components satisfy the conditions $\alpha_j > -1$ $(j = 1, \dots, n)$. We introduce the following norms:

$$\|f\|_{L_p(E_s^{+n}, \vec{\alpha})} = \left(\int_{E_s^{+n}} \prod_{j=1}^{n} y_j^{\alpha_j} \, |f(\mathbf{y})|^p \, d\mathbf{y} \right)^{1/p},$$

$$\|f\|_{L_p^{(\mathbf{r})}(E_s^{+n}, \vec{\alpha})} = \sum_{i=1}^{s} \left\| D_{(i)}^{|\mathbf{r}^{(i)}|} f \right\|_{L_p(E_s^{+n}, \vec{\alpha})},$$

where

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$$D_{(i)}^{|\mathbf{r}^{(i)}|} f(\mathbf{y}) = \frac{\partial^{|\mathbf{r}^{(i)}|} f(\mathbf{y})}{(\partial y_1^{(i)})^{r_1^{(i)}} \cdots (\partial y_{n_i}^{(i)})^{r_{n_i}{}^{(i)}}}, \qquad |\mathbf{r}^{(i)}| = \sum_{j_i=1}^{n_i} r_{j_i}^{(i)},$$

$$\|f\|_{W_p^{(\mathbf{r})}(E_s^{+n},\vec{\alpha})} = \|f\|_{L_p(E_s^{+n},\vec{\alpha})} + \|f\|_{L_p^{(\mathbf{r})}(E_s^{+n},\vec{\alpha})}.$$

Definition. The spaces $L_p(E_s^{+n},\vec{\alpha})$, $L_p^{(\mathbf{r})}(E_s^{+n},\vec{\alpha})$, and $B_p^{(\mathbf{r})}(E_s^{+n},\vec{\alpha})$ shall be called the closures of the set of smooth functions finite in E^n, respectively in the norms

$$\|f\|_{L_p(E_s^{+n},\vec{\alpha})}, \qquad \|f\|_{L_p^{(\mathbf{r})}(E_s^{+n},\vec{\alpha})}$$

and

$$\|f\|_{B_p^{(\mathbf{r})}(E_s^{+n},\vec{\alpha})}.$$

3. Let $\varphi(\mathbf{y})$ be a smooth function defined in E^n. Suppose that numbers $\rho_j > 0$ $(j = 1, \dots, n)$ are given. For each j set $\rho_j = \bar{\rho}_j + \beta_j$, where $\bar{\rho}_j$ is the greatest integer less than ρ_j, so that $0 < \beta_j \le 1$. Introduce

norms

$$\|\varphi\|_{\mathcal{L}_{p,y_j}^{\rho_j}(E^{+n},\vec{\alpha})} = \left(\int_0^\infty \|\Delta_j^{1+[\beta_j]}(t) D_j^{\bar{\rho}_j} \varphi\|_{L_p(E^{+n},\vec{\alpha})} \frac{dt}{t^{1+p\beta_j}} \right)^{1/p},$$

where $\Delta_j^{1+[\beta_j]}(t)\varphi$ is the finite difference of order $1 + [\beta_j]$ with respect to the variable y_j with step t; $[\beta_j]$ is the integer part of β_j, $D_j^{\bar{\rho}_j}\varphi(y) = \partial^{\rho_j}\varphi(y)/\partial y_i^{\bar{\rho}_j}$,

$$\|\varphi\|_{\mathcal{L}_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})} = \sum_{i=1}^{n} \|\varphi\|_{\mathcal{L}_{p,y_i}^{\rho_i}(E^{+n},\vec{\alpha})},$$

$$\|\varphi\|_{B_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})} = \|\varphi\|_{L_p(E^{+n},\vec{\alpha})} + \|\varphi\|_{\mathcal{L}_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})}.$$

Definition. By the spaces $\mathcal{L}_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})$ and $B_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})$ we shall mean the closure of the set of smooth finite functions in E^n, respectively in the norms $\|\varphi\|_{\mathcal{L}_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})}$ and $\|\varphi\|_{B_p^{(\vec{\rho})}(E^{+n},\vec{\alpha})}$.

For $\alpha_i = 0$ $(i = 1, \dots, n)$ these spaces are denoted respectively by $\mathcal{L}_p^{(\vec{\rho})}(E^{+n})$ and $B_p^{(\vec{\rho})}(E^{+n})$.

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4. Some embedding theorems have been obtained for the spaces defined above; these are a development of the corresponding results of works (1-11).

Theorem 1. Suppose:

1) $p > 1$, $\mathbf{r} = (\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(s)}) = (r_1^{(1)}, \dots, r_{n_1}^{(1)}; \dots; r_1^{(s)}, \dots,$

$$\dots, r_{n_s}^{(s)}) = (r_1, \dots, r_n)$$

is a vector with integer positive components;

2) $\vec{\nu} = (\vec{\nu}^{(1)}, \dots, \vec{\nu}^{(s)}) = (\nu_1^{(1)}, \dots, \nu_{n_1}^{(1)}; \dots; \nu_1^{(s)}, \dots, \nu_{n_s}^{(s)}) = (\nu_1, \dots,$

$$\dots, \nu_n)$$

is a vector with integer nonnegative components; 3) m is a natural number such that $0 < m \le n$; 4) $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\vec{\gamma} = (\gamma_1, \dots, \gamma_m)$ are vectors whose components satisfy the conditions

$$\alpha_j \ge \gamma_j \ge 0 \quad (j = 1, \dots, m), \qquad \alpha_\eta > -1 \quad (\eta = m+1, \dots, n);$$

3)

$$\varepsilon = \min_{i \in \{1+, \dots, s\}} \{n_i\} - \frac{1}{p} \sum_{j=m+1}^{n} \frac{1}{r_j} - \sum_{j=1}^{n} \frac{\nu_j}{r_j} - \frac{1}{p} \sum_{j=1}^{n} \frac{\alpha_j}{r_j} + \frac{1}{p} \sum_{j=1}^{m} \frac{\gamma_j}{r_j} > 0;$$

4) $f(y) \in W_p^{(\mathbf{r})}(E_s^{+n}, \alpha)$. Then

$$\Psi \in L_p(E^{+m}, \vec{\gamma}), \quad \text{where } \Psi = D_{(1)}^{|\vec{\nu}^{(1)}|} \cdots D_{(s)}^{|\vec{\nu}^{(s)}|} f(y_1, \dots, y_m, 0, \dots, 0),$$

and the inequality

$$\|\Psi\|_{L_p(E^{+m}, \vec{\gamma})} \le c \, \|f\|_{W_p^{(\mathbf{r})}(E_s^{+n}, \vec{\alpha})}$$

holds, where c is a constant independent of f.

Theorem 2. Under the hypotheses of Theorem 1, if all n_i $(i = 1, \dots, s)$ are equal to one another, then the inequality

$$\|\Psi\|_{L_p(E^{+m}, \vec{\gamma})} \le c \, \|f\|_{L_p(E_s^{+n}, \vec{\alpha})}^{\varepsilon/(\varepsilon+\delta)} \|f\|_{L_p^{(\mathbf{r})}(E_s^{+n}, \vec{\alpha})}^{\delta/(\varepsilon+\delta)}$$

holds, where

$$\delta = \frac{1}{p} \sum_{j=m+1}^{n} \frac{1}{r_j} + \sum_{j=1}^{n} \frac{\nu_j}{r_j} + \frac{1}{p} \sum_{j=1}^{n} \frac{\alpha_j}{r_j} - \frac{1}{p} \sum_{j=1}^{m} \frac{\gamma_j}{r_j} > 0.$$

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Theorem 3. Let the conditions 1)-6) of Theorem 1 be satisfied and, in addition, let one of the following conditions be satisfied:

I. $0 < \rho_j \leqslant \varepsilon r_j$ $(j = 1, \dots, m)$, if $m < n$. II. $0 < \rho_j < \varepsilon r_j$ $(j = 1, \dots, m)$, if $m \leqslant n$. Then $\Psi \in \mathcal{L}_p^{(\vec{\rho})}(E^{+m}, \vec{\gamma})$, and the inequality

$$\|\Psi\|_{\mathcal{L}_p^{(\vec{\rho})}(E^{+m},\vec{\gamma})} \leqslant c\|f\|_{W_p^{(r)}(E_s^{+n},\vec{\alpha})}$$

holds; c is a constant independent of f.

From Theorems 1 and 3 the following assertion follows:

Theorem 4. Under the conditions of Theorem 3

$$\Psi \in B_p^{(\vec{\rho})}(E^{+m}, \vec{\gamma}),$$

and the inequality

$$\|\Psi\|_{B_p^{(\vec{\rho})}(E^{+m},\vec{\gamma})} \leqslant c\|f\|_{W_p^{(r)}(E_s^{+n},\vec{\alpha})}$$

holds, where c is a constant independent of f.

Theorem 5. Suppose that all the conditions of Theorem 1 are satisfied, $m < n$; moreover, $\rho_j = \varepsilon r_j$ $(j = 1, \dots, m)$, and all n_i $(i = 1, \dots, s)$ are equal to one another. Then the inequality

$$\|\Psi\|_{\mathcal{L}_p^{(\vec{\rho})}(E^{+m},\vec{\gamma})} \leqslant c\|f\|_{L_p^{(r)}(E_s^{+n},\vec{\alpha})}$$

holds.

Theorem 6. Suppose that all the conditions of Theorem 1 hold and that ρ_j $(j = 1, \dots, m)$ are positive integers such that $\rho_j < \varepsilon r_j$ $(j = 1, \dots, m)$. Then

$$\Psi \in L_p^{(\vec{\rho})}(E^{+m}, \vec{\gamma})$$

and the inequality

$$\|\Psi\|_{L_p^{(\vec{\rho})}(E^{+m},\vec{\gamma})} \leqslant c\|f\|_{W_p^{(r)}(E_s^{+n},\vec{\alpha})}$$

holds, where c is a constant independent of f.

From Theorems 1 and 6 the following assertion follows:

Theorem 7. Under the conditions of Theorem 6

$$\Psi \in W_p^{(\vec{\rho})}(E^{+m}, \vec{\gamma}),$$

and the inequality

$$\|\Psi\|_{W_p^{(\vec{\rho})}(E^{+m},\vec{\gamma})} \leqslant c\|f\|_{W_p^{(r)}(E_s^{+n},\vec{\alpha})}$$

holds; where c is a constant independent of f.

5. Some theorems of inverse character have also been obtained.

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Theorem 8. Let $p \geqslant 1$; ν be a natural number; r_i $(i = 1, \dots, n)$ be positive integers; $\alpha_j \geqslant 0$ $(j = 1, \dots, n-1)$, $\alpha_n > -1$. It is assumed that

$$\varepsilon = 1 - \nu/r_n - (1+\alpha_n)/pr_n > 0, \quad \rho_j = \varepsilon r_j \quad (j = 1, \dots, n-1).$$

Then: 1) if on the hyperplane $y_n = 0$ a function

$$\varphi_1(y_1, \dots, y_{n-1}) \in \mathcal{L}_p^{(\vec{\rho})}(E^{+\,n-1}, \vec{\alpha}^{(n-1)})$$

is given, then in the space E^{+n} one can construct a function

$$f_1(y_1, \dots, y_n) \in L_p^{(r)}(E^{+n}, \vec{\alpha})$$

such that

$$D_n^{\nu} f_1(y)\big|_{y_n=0} = \varphi_1(y_1, \dots, y_{n-1}),$$

$$\|f_1\|_{L_p^{(r)}(E^{+n},\vec{\alpha})} \leqslant c\|\varphi_1\|_{\mathcal{L}_p^{(\vec{\rho})}(E^{+\,n-1},\vec{\alpha}^{(n-1)})};$$

2) if on the hyperplane $y_n = 0$ a function $\varphi_2(y_1, \dots, y_{n-1}) \in B_p^{(\vec{\rho})}(E^{+n-1}, \vec{\alpha}^{(n-1)})$ is given, then in Ω one can construct a function $f_2(y_1, \dots, y_n) \in W_p^{(r)}(\Omega, \vec{\alpha})$ such that

$$D_n^{\nu} f_2(y)\big|_{y_n=0} = \varphi_2(y_1, \dots, y_{n-1}),$$

$$\|f_2\|_{W_p^{(r)}(\Omega,\vec{\alpha})} \leq c\|\varphi_2\|_{B_p^{(\vec{\rho})}(E^{+n-1},\vec{\alpha}^{(n-1)})},$$

where c is a constant independent of φ_1 and φ_2,

$$\Omega = \{y \in E^{+n},\ y_1 < \infty\}, \qquad \vec{\alpha}^{(n-1)} = (\alpha_1, \dots, \alpha_{n-1}).$$

Theorem 9. Let $p \geq 1$, $0 < m < n$, ν_j $(j = m+1, \dots, n)$ be nonnegative integers, and $a_n > -1$. Suppose that

$$\varepsilon^* = 1 - \sum_{j=m+1}^{n} \frac{\nu_j}{r_j} - \frac{1}{p}\sum_{j=m+1}^{n} \frac{1}{r_j} - \frac{\alpha_n}{pr_n} > 0,$$

$$\rho_j = \varepsilon^* r_j \quad (j = 1, \dots, m).$$

Then: 1) if on the hyperplane $y_{m+1} = \dots = y_n = 0$ a function $\varphi_1(y_1, \dots, y_m) \in \mathcal{L}_p^{(\vec{\rho})}(E^m)$ is given, then in the space E_n^{+*} one can construct a function $f_1(y_1, \dots, y_n) \in L_p^{(r)}(E_n^{+*}, a_n)$ such that

$$D_{m+1}^{\nu_{m+1}} \cdots D_n^{\nu_n} f_1(y)\big|_{y_{m+1}=0,\dots,y_n=0} = \varphi_1(y_1, \dots, y_m),$$

$$\|f_1\|_{L_p^{(r)}(E_n^{+*},a_n)} \leq c\|\varphi_1\|_{\mathcal{L}p^{(\vec{\rho})}(E^m)};$$

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2) if on the hyperplane $y_{m+1} = \cdots = y_n = 0$ a function $\varphi_2(y_1, \dots, y_m) \in B_p^{(\bar{\rho})}(E^m)$ is given, then in Ω^* one can construct a function $f_2(y_1, \dots, y_n) \in W_p^{(r)}(\Omega^*, a_n)$ such that

$$D_{m+1}^{\nu_{m+1}} \cdots D_n^{\nu_n} f_2(y)\big|_{y_{m+1}=0,\dots,y_n=0} = \varphi_2(y_1, \dots, y_m),$$

$$\|f_2\|_{W_p^{(r)}(\Omega^*, a_n)} \leq c\|\varphi_2\|_{B_p^{(\bar{\rho})}(E^m)},$$

where c is a constant independent of φ_1, φ_2,

$$E_n^{+*} = \{y \in E^n,\, y_n > 0\}, \qquad \Omega^* = \{y \in E_n^{+*},\ |y_i| < \infty$$

$$(i = m+1, \dots, n)\}.$$

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Note: Figure translations are in progress. See original paper for figures.

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