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**Abstract**

**Full Text**

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## MONOTONE NONLINEAR OPERATORS IN BANACH SPACES

*(Presented by Academician P. S. Novikov, 13 I 1965)*

**1.**

Let  $E_1$  and  $E_2$  be real or complex Banach spaces. Consider a linear or nonlinear operator  $F(x)$ , defined on a set  $D \subset E_1$ , with values in the space  $E_2$ . Let  $B$  be a linear continuous operator mapping the space  $E_2$  into the space  $E_1^*$ , conjugate to  $E_1$ , and having an inverse.

**Definition 1.** The operator  $F(x)$  is called  **$B$ -monotone** on the set  $D$  if, for any  $x_2, x_1 \in D$ , the inequality

$$\operatorname{Re}(BF(x_2) - BF(x_1), x_2 - x_1) \geq 0 \quad (1)$$

holds, where the notation  $(z, x)$  denotes the value of the linear functional  $z \in E_1^*$  on the element  $x \in E_1$ .

In particular, if  $E_2 = E_1^*$  and  $B = I$  is the identity operator in the space  $E_1^*$ , the operator  $F(x)$  will be called  **$I$ -monotone**. For the case of real Banach spaces, the concept of an  $I$ -monotone operator was introduced in 1960 by the author in paper <sup>(3)</sup>. Later, in paper <sup>(13)</sup>, this concept was generalized by G. J. Minty\* to operators defined in a complex Banach space with values in its conjugate space (in this connection see Remark 1 of the present paper). In the cited papers, some properties of  $I$ -monotone operators were considered.  $I$ -monotone operators have already found fairly broad application.

Let us first of all mention new fixed-point theorems for equations with  $I$ -monotone operators, established in the works of F. E. Browder, M. M. Vainberg, M. A. Krasnosel'skii, R. I. Kachurovskii, G. J. Minty, and other authors. Monotone operators have been applied to the study of minima of functionals in Banach spaces (see <sup>(4, 6)</sup> and others), and in considering iteration processes for equations in Banach spaces <sup>(5, 7)</sup>. With the help of monotone operators one can prove the existence of solutions of nonlinear integral equations and of systems of algebraic equations. Finally, recently there has appeared a large cycle of papers by F. E. Browder in which monotone operators are used to investigate boundary-value problems for partial differential equations of elliptic, hyperbolic, and parabolic types. Among other works using monotone operators,

we mention papers (8, 15, 16). However, only operators acting from the space  $E_1$  into  $E_1^*$  can be  $I$ -monotone. In the present paper a further generalization of the concept of monotonicity is given to operators not necessarily acting into their conjugate space, and properties of such operators are considered.

In what follows, unless otherwise specified, it will be understood that the spaces  $E_1$  and  $E_2$  may be either real or complex.

\* Apparently, G. J. Minty was not acquainted with the author's paper (3), since references are absent.

**2.** First of all let us note several simple propositions giving necessary and sufficient conditions for the  $B$ -monotonicity of the operator  $F(x)$ .

Let  $C$  be a linear operator acting from the space  $E_1$  into  $E_2$ ; let  $B$  be a linear operator acting from the space  $E_2$  into  $E_1^*$ . We shall call the operator  $C$   $B$ -positive if, for any  $h \in E_1$ , the inequality

$$\operatorname{Re}(BC h, h) \geq 0$$

holds.

**Lemma 1.** *Let an operator  $F(x)$  with values in  $E_2$  be given on a convex open set  $D \subset E_1$ , and let it have a Gateaux derivative  $F'(x)$  at every point of the set  $D$ . Suppose that the function of the variable  $t$*

$$\operatorname{Re}(BF'(x + th)h, h)$$

*is continuous in  $t$  at the point  $t = 0$  for every fixed  $x \in D$ ,  $h \in E_1$ . In order that the operator  $F(x)$  be  $B$ -monotone on the set  $D$ , it is necessary and sufficient that its Gateaux derivative  $F'(x)$  be a  $B$ -positive operator at every point  $x \in D$ .*

**Lemma 2.** *Let the operator  $F(x)$  be defined on a convex set  $D \subset E_1$ . In order that the operator  $F(x)$  be  $B$ -monotone on the set  $D$ , it is necessary and sufficient that for each  $x \in D$  there exist a sphere  $S(x)$  with center at the point  $x$  such that  $F(x)$  is  $B$ -monotone on  $D \cap S(x)$ .*

**Lemma 3.** *Let an operator  $F(x)$  be given on a convex set  $D \subset E_1$ . For the  $B$ -monotonicity of the operator  $F(x)$  on the set  $D$  it is necessary and sufficient that the function of the real variable  $t$ ,  $0 \leq t \leq 1$ ,*

$$\operatorname{Re}(BF(x + th), h)$$

*be nondecreasing for any fixed  $x$ ,  $(x + h) \in D$ .*

As a consequence of this one obtains

**Lemma 4.** *Let an operator  $F(x)$  be given on an open convex set  $D \subset E_1$ . For the  $B$ -monotonicity of the operator  $F(x)$  on the set  $D$  it is necessary and sufficient that, for any fixed  $x$ ,  $(x + h) \in D$ , at the point  $t = 0$  the inequality*

$$\frac{d}{dt} \operatorname{Re}(BF(x + th), h) \geq 0$$

hold, where  $t$  is real.

Lemma 1 generalizes the corresponding proposition from (3), Lemma 2–Theorem 5 from (13), and Lemma 4–Theorem 6 from (13), which gives only a sufficient condition for  $I$ -monotonicity.

If, for any  $x_2, x_1 \in D$ ,  $x_2 \neq x_1$ , a strict inequality holds in relation (1), then  $F(x)$  is called a **strictly  $B$ -monotone operator**.

**Lemma 5.** *A strictly  $B$ -monotone operator has an inverse, which is a  $B^*$ -monotone operator ( $B^*$  is the operator adjoint to  $B$ ).*

The following theorem on the norm of the values of a  $B$ -monotone operator holds.

**Theorem 1\*.** *Let  $D$  be a bounded convex closed set of a Banach space  $E_1$ . Let a  $B$ -monotone operator  $F(x)$  with values in a Banach space  $E_2$  be given on the set  $D$ , and let it be bounded on the boundary  $\Gamma_D$  of the set  $D$  (i.e.*

$$\sup_{x \in \Gamma_D} \|F(x)\| < \infty).$$

*Then  $F(x)$  is bounded on the whole set  $D$  (i.e.*

$$\sup_{x \in D} \|F(x)\| < \infty).$$

)

In particular, this proposition is also valid for  $I$ -monotone operators.

In the work (3) functionals are considered whose gradients are  $I$ -monotone operators. For such functionals there also holds the following

**Theorem 2\*.** *If on a bounded closed convex set  $D$  of a real Banach space  $E_1$  there is given a real functional  $f(x)$ , whose gradient is an  $I$ -monotone operator*

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\* In the proof of Theorems 1 and 2 it was assumed that  $D$  contains at least one interior point.

in  $D$ , and the gradient is bounded on the boundary  $\Gamma_D$  of the set  $D$ , then the functional  $f(x)$  is continuous at every interior point of the set  $D$  and is bounded on the set  $D$  (i.e.,  $\sup_{x \in D} |f(x)| < \infty$ ).

3. An operator  $F(x)$  from  $E_1$  into  $E_2$  (cf. (12,14)) is called **semicontinuous** on the set  $D \subset E_1$  if it maps every strongly convergent sequence from  $D$  into a weakly convergent sequence from  $E_2$ . Let  $x_0$  be some interior point of the domain  $D$ ; let  $S$  be a surface lying in the domain  $D$  and enclosing the point  $x_0$  (this means that every straight line passing through the point  $x_0$  intersects the surface  $S$  in exactly two points lying on different sides of  $x_0$ ). Such a surface  $S$  will be called **admissible**. We formulate one theorem on a fixed point of a  $B$ -monotone operator.

**Theorem 3.** Let  $D$  be an open convex bounded set of the reflexive Banach space  $E_1$ ; let  $F(x)$  be a linear or nonlinear semicontinuous  $B$ -monotone operator, defined on the set  $D$  with values in the Banach space  $E_2$ . Let the surface  $S$ , inside which there lies some interior point  $x_0$  of the domain  $D$ , be admissible, and suppose that for all  $z \in S$  the condition

$$\operatorname{Re}(BF(z), z - x_0) \geq 0$$

is fulfilled. Then there exists at least one point  $x' \in D$  such that  $F(x') = \theta$ .

The result established in this theorem intersects with the results of F. E. Browder, M. M. Vainberg, M. A. Krasnosel'skii, G. J. Minty, and the author. We give one consequence of Theorem 3.

**Theorem 4.** Let the domain  $D$  and the surface  $S$  satisfy the conditions of Theorem 3. Let there be given in the domain  $D$  a semicontinuous Gâteaux-differentiable operator  $F(x)$  with values in the Banach space  $E_2$ ; let  $A$  be a linear continuous operator from  $E_1$  onto  $E_2^*$ , having a continuous inverse.

If:

- 1) for all  $x \in D$ ,  $h \in E_1$ 

$$\operatorname{Re}(F'(x)h, Ah) \geq 0;$$
- 2)  $\operatorname{Re}(F(z), A(z - x)) \geq 0$  for all  $z \in S$ ,

then there exists at least one solution of the equation  $F(x) = \theta$  in the domain  $D$ .

This theorem has considerable advantages in comparison with the corresponding theorem from (2), proved by M. A. Krasnosel'skii.

4. We now make some remarks.

**Remark 1.** Monotonicity in complex spaces can be defined by means of monotonicity in real spaces. Indeed, let  $F(x)$  be an operator acting from the complex Banach space  $E_1$  into the conjugate space  $E_1^*$ . Put  $x = u + iv$ ,

$$F(x) = F(u + iv) = F_1(u, v) + iF_2(u, v),$$

where  $u, v$  are elements of the real space  $E_1$ ;  $F_1(u, v)$  and  $F_2(u, v)$  are elements of the real space  $E_1^*$ . Let  $X_1$  be the direct sum of two real spaces  $E_1$ ; let  $X_1^*$  be its conjugate. Consider the operator  $\Phi(x)$ , defined in the real space  $X_1$  with values in  $X_1^*$ :

$$\Phi(x) = \Phi(\{u, v\}) = \{F_1(u, v); F_2(u, v)\}.$$

Then

$$(\Phi(x_2) - \Phi(x_1), x_2 - x_1) = \operatorname{Re}(F(x_2) - F(x_1), x_2 - x_1),$$

and therefore one can give the following definition: the operator  $F(x)$  is called  **$I$ -monotone in the complex space  $E_1$**  if the operator  $\Phi(x)$  is  $I$ -monotone in the real space  $X_1$ . The situation is analogous with  $B$ -monotonicity.

**Remark 2.** Along with Definition 1, one can give another definition of monotonicity. Let  $A$  be a linear continuous operator from  $E_1$  onto  $E_2^*$ , having an inverse. The operator  $F(x)$ , defined on the set  $D \subset E_1$  with values in  $E_2$ , may be called  **$A$ -monotone** on

on the set  $D$ , if

$$\operatorname{Re}(F(x_2) - F(x_1), Ax_2 - Ax_1) \geq 0$$

for any  $x_2, x_1 \in D$ . We note that the operator  $F(x)$  will in this case be  $A^*$ -monotone in the sense of Definition 1.

**Remark 3.** Monotonicity in the sense of Definition 1 and Remark 2 could also have been defined with the aid of nonlinear operators  $B$  and  $A$ , satisfying certain conditions.

**Remark 4.** Monotone operators can be considered in locally convex spaces.

**Remark 5.** Following Minty, who considered  $M$ -related sets, one can consider  $M_B$ -related sets in the space  $(E_1 + E_2)$ . Pairs  $\{x, y\}$  and  $\{x', y'\}$  from the space  $E_1 + E_2$  are called  $M_B$ -related if

$$\operatorname{Re}(By' - By, x' - x) \geq 0.$$

A set  $D_0 \subset E_1 + E_2$  is called totally  $M_B$ -related if every pair of points from  $D_0$  is  $M_B$ -related.

**Lemma 6** (generalized Minty lemma). *Let a set  $D \in E_1$  contain the point  $x_0$  and densely surround  $x_0$ . Let the operator  $F(x) : D \rightarrow E_2$  be ray-continuous at the point  $x_0$ . Let the operator  $B$  satisfy the conditions of Definition 1. If the point  $(x_0, y_0)$  is  $M_B$ -related to every point of the graph of the operator  $F(x)$ , then  $y_0 = F(x_0)$ .*

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*Note: Figure translations are in progress. See original paper for figures.*

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