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Abstract

Full Text

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TOPOLOGICAL INVARIANCE OF RATIONAL PONTRYAGIN CLASSES

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The aim of the present work is to prove the theorem:

Theorem 1. *Let two smooth or PL-manifolds M_1^n, M_2^n and a continuous homeomorphism $h : M_1^n \to M_2^n$ be given.*

Then $h^* p_i(M_2^n) = p_i(M_1^n)$, where $p_i(M^n)$ are the Pontryagin classes of the manifold M^n with rational or real coefficients.

In the present note we shall give an outline of the proof of this theorem for smooth simply connected manifolds M_1^n and M_2^n; the case of non-simply connected manifolds reduces to the simply connected one. Let us note that the method of the author' s preceding work ([2]), where this fact was established in a special case, differs from that proposed here, although it is in close conceptual connection with it.

Theorem 1 follows from the following basic lemma:

Basic lemma. *Suppose that on the direct product $M^{4k} \times R^m$ some smooth structure is given, turning $M^{4k} \times R^m$ into a smooth open manifold W; here M^{4k} is a compact closed simply connected manifold.*

Then the formula holds:

$$(L_k(W), [M^{4k}] \otimes 1) = \tau(M^{4k}),$$

where L_k is the Hirzebruch polynomial in the Pontryagin classes of the manifold W; $\tau(M^{4k})$ is the signature of the manifold M^{4k}.

The proof is divided into several steps.

Step 1. Choose in R^m an open subset $T^{m-1} \times R \subset R^m$, where T^{m-1} is an $(m-1)$-dimensional torus. Consider the open submanifold

$$W_1 = M^{4k} \times T^{m-1} \times R \subset W.$$

The following simple lemma holds.

Lemma 1. *There exists a smooth closed submanifold $V_1 \subset W_1$, realizing the cycle $[M^{4k} \times T^{m-1}] \otimes 1 \in H_{4k+m-1}(W_1)$, and such that the natural projection $p : V_1 \to M^{4k} \times T^{m-1}$ induces an isomorphism of homotopy groups and homology groups in dimensions $< 2k + [(m-1)/2]$.*

Similarly, let W be a smooth manifold of the homotopy type $M^{4k} \times T^m$, and let W_1' be its covering of homotopy type $M^{4k} \times T^{m-1}$. There is a natural projection of degree $+1$, $f : W_1' \to M^{4k} \times T^{m-1}$, and $H_{4k+m-1}(W_1') = Z$.

Lemma 1′. *There exists a smooth submanifold $V_1' \subset W_1'$, realizing the basic cycle of the group $H_{4k+m-1}(W_1') = Z$, and such that the natural projection $f|V_1' \to M^{4k} \times T^{m-1}$ induces an isomorphism of homotopy groups and homology groups in dimensions $< 2k + [(m-1)/2]$.*

The proof of both lemmas is analogous. It is carried out by successive Morse surgeries on the kernel of the embedding $V_1 \subset W_1$ or $V_1' \subset W_1'$, approximately analogously to the works ([1],[3]). Here it should be noted additionally that, in view of the nontriviality of the group ring of an abelian group, all kernels

the mapping f in homotopy are finitely generated as modules over $Z(\pi_1)$, $\pi_1 = Z + \cdots + Z$. Thus, in the indicated dimensions we can kill the kernels of the mapping f in homotopy. We note that on the universal coverings the covering mapping $\hat{f} : V_1 \to M^{4k} \times R^{m-1}$ has degree $+1$ as a proper mapping. Therefore the formula $f_* \hat{D} f^* D = 1$ holds, where D is Poincaré duality. In addition, in our case the following equality is not hard to prove:

$$\operatorname{Ker} f_*(\pi_i)/Z_0(\pi_1)\operatorname{Ker} \hat{f}_*(\pi_i) = \operatorname{Ker} f_*(H_i),$$

(the Hurewicz theorem), where $\operatorname{Ker} \hat{f}_*(\pi_j) = \operatorname{Ker} f_*(\pi_j) = 0$ for $j < i$, and also $\operatorname{Ker} f_*(\pi_i) = \operatorname{Ker} f_*(H_i)$. From this one can already derive Lemmas 1 and 1′.

Step 2. The study of the dimension $i = [2k + (m-1)/2]$ is more difficult. To this end we shall make some remarks on the intersection index. Suppose there is a covering $P : M \to \hat{M}$ with monodromy group $\pi : \hat{M} \to M$. The group π acts on the groups $H_i(M)$. Consider two cycles $x, y \in H_*(M)$ and all elements $a_i \in \pi$. The equality

$$(p_* x) \circ (p_* y) = \sum_{a_i \in \pi} x \circ (a_i y)$$

holds, where the sign $\circ$ denotes the intersection index respectively in M and $\hat{M}$. Suppose, on the basis of Lemmas 1 and 1′, there is a submanifold V_1 (or V_1') $\subset W_1$ (or W_1') and a projection $W_1 \to M^{4k} \times T^{m-1}$, of degree $+1$ on V_1, which is a homotopy equivalence on W_1 and a homotopy equivalence in dimensions $< [2k + (m-1)/2]$ on V_1. Denote by N the kernel $\operatorname{Ker} f_*(\pi_l)$ for $i = [2k + (m-1)/2]$; then $\operatorname{Ker} f_*(H_l) = N/Z_0(\pi_1)N$, where $\pi_1 = Z + \cdots + Z$. Moreover, an analogous formula holds for the mapping of all intermediate coverings over V_1 and $M^{4k} \times T^{m-1}$ corresponding to subgroups $\pi' \subset \pi_1$. If π' has finite index in π_1, then on the kernel of the mapping in homology of the corresponding coverings there is Poincaré duality of the usual type in view of the properties of mappings of degree +1.

As has already been noted, the intersection index on the homotopy kernels of π'-coverings is obtained from the intersection index on N by the following formula:

$$(qx) \circ (qy) = \sum_{a_i \in \pi'} x \circ (a_i y), \qquad q : N \to N/Z_0(\pi')N.$$

It is also important that in our case the kernels of homotopy mappings on all coverings split into the union of the "right kernels" and "left kernels" from V_1 to W_1 (or from V_1' to W_1').

Now we do the following:

1. For odd values of the number $4k + m - 1$, by surgeries over V_1 inside W_1 (only on one side of V_1) on a $Z(\pi_1)$-basis of the homotopy "right kernel," we arrange that the "right kernel" in homotopy be trivial on V_1 in dimension $l = [2k + (m-1)/2]$. Then, by analogy with (1), the homology kernels in this dimension will be free on all finite-sheeted coverings.

2. For even values $2l = 4k + m - 1$, it is proved that $N/N_\infty = F \oplus \cdots \oplus F$; here N_∞ is the kernel of the projection to all finite-sheeted coverings and

$$F = \sum_{a_i \in \pi'} a_i Q, \qquad \dim Q = 2;$$

the intersection matrix on Q has the form

$$\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$$

and $a_{iQ} \cdot a_{jQ} = 0$ for $a_i \neq a_j$, where π' is a subgroup of some finite index in π_1.

3. Next, by surgeries on basis elements of the $Z(\pi_1)$-module N, in both cases (even and odd $4k+m-1$) we arrange that the kernels of the homotopy mappings on all finite-sheeted coverings are trivial and that the homotopies in dimensions $< l$ are not changed.

4. However, at this stage the surgeries are no longer performed inside the manifold W_1 (or W_1'), but abstractly, on the π'-covering.

It is easy to see that after these operations the scalar product of the class $L_k(V_2)$ with the cycle $f_*^{-1}([M^{4k}] \otimes 1)$ continues to coincide with the scalar product

$$(L_k(W_1), f_*^{-1}([M^{4k}] \otimes 1)),$$

since this cycle lies on V_1, and V_1 has trivial normal bundle in W up to the last surgery.

Step 3. There is an extremely essential fact: as a result of the last surgery we obtain

$$N = \operatorname{Ker} f_*(\pi_1) = 0.$$

This fact is not obvious, since we had only that $N/N_\infty = 0$, where N_∞ is the kernel of the projection onto all finite-sheeted coverings. By the Noetherian property of the ring $Z(\pi_1)$, the module N is finitely generated over $Z(\pi_1)$. The module N_∞ is determined here purely algebraically by virtue of Gurevich' s theorem. By a simple argument the problem is reduced to one-dimensional modules and to the case where $\pi_1 = Z$, and it has the following form: there is a polynomial $P(x)$ with integer coefficients such that $P(0) \neq 0$ and $P(1) = 1$, and for every root of unity $\xi \neq 1$ the value $P(\xi)$ is invertible in the integers of the corresponding cyclotomic field. One must prove that $P(x) = 1$. This fact is proved with the aid of elementary algebraic number theory.* An analogous assertion holds for polynomials mod p.

It follows from this that after surgery over V_1 (or V_1') we shall have $\operatorname{Ker} f_*(\pi_1) = 0$.

Therefore the "surgered" map

$$f : V_1 \to M^{4k} \times T^{m-1}$$

will be a homotopy equivalence in dimensions

$$\leq [2k + (m-1)/2].$$

It can be shown that this map f will be a homotopy equivalence in all dimensions on the basis of Poincaré duality, Gurevich' s theorem, and the indicated fact about polynomials mod p.

Step 4. We can now derive our main lemma. Consider

$$W = M^{4k} \times T^{m-1} \times R$$

in some smoothness class, and, on the basis of the preceding, construct a manifold V_1 of homotopy type $M^{4k} \times T^{m-1}$ with the same class $L_k(V_1)$ as $L_k(W)$.

Next, consider the covering over V_1 with group Z, $V_1 = W_2 \to V_1$, and the map

$$f_2 : V_1 \to M^{4k} \times T^{m-2}.$$

On the basis of the preceding, using $V_1 = W_2$, we find an analogous manifold V_2 with the same class $L_k(V_2)$ and a map

$$V_2 \to M^{4k} \times T^{m-2}$$

of degree $+1$, etc., until we reach dimension $4k$. We obtain a manifold V_m of dimension $4k$, of homotopy type M^{4k}, and with the same class $L_k(V_m)$ as

$L_k(W)$, more precisely, with the same scalar product of the class L_k with the corresponding cycle. On the basis of Hirzebruch' s formula we conclude that

$$L_k(V_m) = (L_k(W), f_*^{-1}([M^{4k}] \otimes 1)) = \tau(M^{4k}).$$

The main lemma follows from this.

Remark 1. The author' s method does not yet make it possible to determine the Pontryagin classes of topological microbundles and topological manifolds, since all the arguments were connected with a smooth structure; this seems to be the first question following from the result of this work.

Remark 2. The proof of invariance seems to the author artificial, since Morse surgeries apparently are not essential here. After clarifying the relations between the classes L_k and coverings, it should take a more explicit form, as is shown in the special case by the proof in paper (2) and by the as yet unpublished results of the author and V. A. Rokhlin on these questions.

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Note: Figure translations are in progress. See original paper for figures.