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Abstract

Full Text

MATHEMATICS

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ON UNITARY REPRESENTATIONS OF LOCALLY BICOMPACT GROUPS IN THE SPACE Π_1

(Presented by Academician I. M. Vinogradov, 2 VII 1964)

1. Let G be a locally bicomact group with a countable base of neighborhoods at infinity (in what follows, without further qualification, only such groups are considered), and let $g \rightarrow U_g$ be a continuous unitary representation of the group G in a separable Hilbert space \mathfrak{H} . As is known (see, for example, ⁽¹⁾ or ⁽²⁾, Ch. VIII), every such representation can be realized as a direct integral of continuous irreducible unitary representations of the group G ; this realization is obtained by decomposing the space \mathfrak{H} into a direct integral of Hilbert spaces with respect to a maximal commutative symmetric subalgebra of the algebra of all bounded linear operators commuting with all the operators U_g , $g \in G$.

If the representation is nonunitary, then an analogous fact, in general, no longer holds; even in a finite-dimensional space there may exist reducible, but not completely reducible, representations.

The purpose of this article is to study the structure of representations that are unitary with respect to some indefinite scalar product. For simplicity, we consider the case of a representation unitary (with respect to an indefinite scalar product) in the space Π_1 ; however, the main results remain valid (with the appropriate changes) for unitary representations of groups and symmetric representations of separable symmetric algebras in an arbitrary space Π_ν , $\nu < \infty^*$ (for the definition and basic properties of the spaces Π_ν , see ⁽⁴⁾).

2. Let $g \rightarrow U_g$ be a continuous representation of the group G in the separable space Π_1 , unitary with respect to the indefinite scalar product (ξ, η) in Π_1 ; let M be the totality of all bounded linear operators in Π_1 commuting with all the operators U_g , $g \in G$; and let \mathfrak{A} be some maximal commutative subalgebra in M . By the separability of Π_1 , in \mathfrak{A} there exists a sequence $\{A^{(n)}\}$ dense in \mathfrak{A} in the strong operator topology. Let R be the norm-closed symmetric algebra of operators generated by the operators $A^{(n)}$ and 1. Then $R \subset \mathfrak{A}$, and \mathfrak{A} is the closure of the algebra R in the strong (and weak) operator topology. We apply to R the results of the article ⁽⁵⁾ (here, in the main, the notation of that article is retained). The realization of the algebra R obtained in ⁽⁵⁾ (we shall call it

canonical) generates a realization of the representation $g \rightarrow U_g$, which we shall call the **corresponding** canonical realization of the algebra R .

It follows from (6) that in Π_1 there exists a nonnegative subspace \mathfrak{R} , invariant with respect to all $A \in R$ (and hence also $A \in \mathfrak{A}$); this means that

$$A\xi = \lambda(A)\xi \quad \text{for all } A \in \mathfrak{A}, \xi \in \mathfrak{R}, \quad (2,1)$$

where $A \rightarrow \lambda(A)$ is a homomorphism of the algebra \mathfrak{A} into the field of complex numbers.

* Unitary representations of the Lorentz group in the space Π_κ have recently been investigated by R. S. Ismagilov (3).

According to (5), only the following three cases are possible:

I. There exists a positive subspace \mathfrak{N} , invariant with respect to all operators $A \in \mathfrak{A}$. In this case $\lambda(A)$ does not depend on the choice of the positive subspace \mathfrak{N} .

II. There exists exactly one pair of one-dimensional skew-conjugate null subspaces $\mathfrak{N}, \mathfrak{N}'$, invariant with respect to all $A \in \mathfrak{A}$, and

$$A\xi = \lambda(A)\xi, \quad A\eta = \overline{\lambda(A^*)}\eta \quad (2,2)$$

for $A \in \mathfrak{A}$, $\xi \in \mathfrak{N}$, $\eta \in \mathfrak{N}'$.

III. There exists exactly one nonnegative one-dimensional subspace \mathfrak{N} , invariant with respect to all $A \in \mathfrak{A}$; this subspace is null, and

$$\lambda(A^*) = \overline{\lambda(A)} \quad \text{for all } A \in R.$$

3. Suppose case I occurs. Put

$$\mathfrak{H}^{(0)} = \{\xi : \xi \in \Pi_1, A\xi = \lambda(A)\xi \text{ for all } A \in \mathfrak{A}\}, \quad \mathfrak{H}^{(1)} = \mathfrak{H}^{(0)\perp}; \quad (3,1)$$

then $\mathfrak{H}^{(0)}$ is one-dimensional or of type Π_1 , $\mathfrak{H}^{(1)}$ is negative, and $\mathfrak{H}^{(0)}$ and $\mathfrak{H}^{(1)}$ are invariant with respect to all U_g , $g \in G$, and also with respect to all $A \in \mathfrak{A}$. Let $U_g^{(0)}, A^{(0)}$ and $U_g^{(1)}, A^{(1)}$ be the restrictions of the operators U_g, A to $\mathfrak{H}^{(0)}$ and $\mathfrak{H}^{(1)}$, respectively; put $\mathfrak{A}^{(j)} = \{A^{(j)}, A \in \mathfrak{A}\}$, $M^{(j)} = \{A^{(j)}, A \in M\}$, $j = 0, 1$. Then $M^{(j)}$ is the set of all bounded linear operators in $\mathfrak{H}^{(j)}$ commuting with all $U_g^{(j)}$, $g \in G$, and $\mathfrak{A}^{(j)}$ is a maximal commutative subalgebra in $M^{(j)}$. We shall call the representation $g \rightarrow U_g$ operator-irreducible (see (7)) if for it M consists only of operators of the form $\lambda 1$. In the case of an ordinary unitary representation, operator irreducibility is equivalent to ordinary irreducibility. In the

space Π_1 , ordinary irreducibility no longer follows from operator irreducibility; however, we shall not here study operator-irreducible unitary representations in Π_1 .

From (3,1) and the maximality of the algebra \mathfrak{A} it follows easily that $M^{(0)}$ consists only of operators of the form $\lambda 1$; consequently, the representation $g \rightarrow U_g^{(0)}$ is operator-irreducible. Taking further into account that $\mathfrak{A}^{(1)}$ is a maximal commutative subalgebra in $M^{(1)}$, we arrive at the following result:

Theorem 1. *In case I, the canonical realization of the algebra R corresponds to a realization of the representation $g \rightarrow U_g$ in the form of an orthogonal sum of a one-dimensional representation or an operator-irreducible representation $g \rightarrow U_g^{(0)}$ in the space Π_1 , and a representation $g \rightarrow U_g^{(1)}$ in a negative space, which is the direct integral*

$$U_g^{(1)} = \int_T U_g(t) d\sigma$$

of representations $g \rightarrow U_g(t)$, continuous unitary in the ordinary sense, irreducible for σ -almost every $t \in T$.

4. Suppose case II occurs. Put $\mathfrak{H}^{(0)} = \mathfrak{N} \dot{+} \mathfrak{N}'$, $\mathfrak{H}^{(1)} = \mathfrak{H}^{(0)\perp}$. Then $\mathfrak{H}^{(0)}$, $\mathfrak{H}^{(1)}$ are invariant with respect to all U_g , $g \in G$, and $A \in \mathfrak{A}$, and one can define $U_g^{(j)}$, $A^{(j)}$, $M^{(j)}$, $\mathfrak{A}^{(j)}$, $j = 0, 1$, as in § 3. In addition, \mathfrak{N} and \mathfrak{N}' are invariant with respect to all operators U_g , and

$$U_g \xi = \tau(g)\xi, \quad U_g \eta = \overline{\tau(g^{-1})}\eta \quad (4.1)$$

for all $g \in G$, $\xi \in \mathfrak{N}$, $\eta \in \mathfrak{N}'$, where $\tau(g)$ is a certain (in general nonunitary) character of the group G . As in § 3, $\mathfrak{H}^{(1)}$ is negative, $M^{(1)}$ is the set of all bounded linear operators in $\mathfrak{H}^{(1)}$ commuting with all operators $U_g^{(1)}$, $g \in G$, and $\mathfrak{A}^{(1)}$ is a maximal commutative subalgebra in $M^{(1)}$. Therefore:

Theorem 2. *In case II, the canonical realization of the algebra R corresponds to a realization of the representation $g \rightarrow U_g$ in the form of an orthogonal sum of a two-dimensional representation in the space $\mathfrak{N} \dot{+} \mathfrak{N}'$ of type Π_1 , defined*

by the formulas (4.1), and the representation $g \rightarrow U_g^{(1)}$ in the negative space, which is the direct integral $U_g^{(1)} = \int_T U_g(t) d\sigma$ of continuous unitary representations $g \rightarrow U_g(t)$ in the usual sense, irreducible for σ -almost every $t \in T$.

5. Finally, suppose case III holds. Then \mathfrak{R} is invariant with respect to all U_g , $g \in G$, and

$$U_g \xi = \tau(g)\xi \quad \text{for } \xi \in \mathfrak{R}, \quad g \in G, \quad (5.1)$$

where $\tau(g)$ is a unitary character of the group G .

Let $T, \sigma, \zeta(t), \mathfrak{P} = \int_{T_1} \mathfrak{H}(t) d\sigma$, \mathfrak{H} be a bicomact space, a Borel measure on T , a vector-function, and negative Hilbert spaces realizing the canonical realization of the algebra R (see (3)), so that $\Pi_1 = (\mathfrak{R} + \mathfrak{R}') \oplus \mathfrak{P} \oplus \mathfrak{H}$, where \mathfrak{R}' is coisotropic with \mathfrak{R} . Here $T_1 = T - \{t_0\}$ (case IIIa), or $T_1 = T$ (case IIIb). Recall that in case IIIb $\mathfrak{H} = (0)$ and one may assume $\zeta(t) \equiv 0$. Let $\xi_0 \in \mathfrak{R}$ and $\eta_0 \in \mathfrak{R}'$ be such that $(\xi_0, \eta_0) = 1$.

Theorem 3. In case IIIa, the canonical realization of the algebra R entails the realization of the representation $g \rightarrow U_g$ by the formulas

$$U_g \xi_0 = \tau(g) \xi_0, \quad (4.2a)$$

$$U_g \{p(t)\} = - \int ((U_g(t) - \tau(g) 1)p(t), \zeta(t)) d\sigma \cdot \xi_0 + \{U_g(t)p(t)\}, \quad (4.2)$$

$$U_g q = (q, q(g)) \xi_0 + U_g(t_0) q, \quad (4.2)$$

$$U_g \eta_0 = \alpha(g) \xi_0 + \tau(g) \eta_0 + \{(U_g(t) - \tau(g)) \zeta(t)\} + q(g^{-1}), \quad (4.2)$$

where $g \rightarrow U_g(t)$ are continuous representations of the group G , irreducible for σ -almost every $t \in T_1$, $\tau(g)$ is a unitary character of the group G , $\alpha(g)$ is a continuous numerical function on G , and $q(g)$ is a continuous vector-function on G with values in \mathfrak{H} . If $\sigma(\{t_0\}) = 0$, then $\mathfrak{H} = (0)$, and formula (4.2) and the last term in (4.2) should be omitted.

In case IIIb the representation is given by the formulas

$$U_g \xi_0 = \tau(g) \xi_0, \quad U_g \{p(t)\} = \{U_g(t)p(t)\}, \quad U_g \eta_0 = \alpha(g) \xi_0 + \tau(g) \eta_0,$$

where $\tau(g)$ is a unitary character of the group G , and $g \rightarrow U_g(t)$ are continuous representations of the group G , irreducible for σ -almost every $t \in T$, i.e., in case IIIb the representation $g \rightarrow U_g$ is the orthogonal sum of a two-dimensional unitary representation in the two-dimensional space Π_1 and the direct integral of ordinary irreducible unitary representations in a negative space.

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