Classes of Motions with Periods in Partially Ordered Dynamical Systems

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Abstract

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Partially ordered dynamical systems were first introduced for consideration by E. A. Barbashin [1, 2]. In contrast to ordinary and general [3] dynamical systems, here, instead of the group of real numbers or another topological group, an arbitrary partially ordered group is considered.

In the present note the concept of a period of a point of the phase space of a partially ordered dynamical system is introduced. Proceeding from the properties of the group of periods, six sets of motions (points) are defined, each of which, for ordinary dynamical systems, coincides with the set of periodic motions, and it is proved that for partially ordered dynamical systems all these classes are distinct.

By a **partially ordered dynamical system** we shall mean a collection $[R, G, f]$, consisting of a metric space R, a nontrivially partially ordered group G [4, 5], and a function f mapping $R \times G$ into R and possessing the following three properties:

1) $f(p, e) = p$, where e is the identity of the group G, and $p \in R$.
2) $f(f(p, g_1), g_2) = f(p, g_1 g_2)$ for arbitrary $g_1, g_2 \in G$, $p \in R$.
3) For any point $p_0 \in R$, element $g_0 \in G$, and $\varepsilon > 0$, there exists a $\delta > 0$ such that, whatever the point $p \in S(p_0, \delta)$, the inequality holds

$$\rho(f(p, g_0),\ f(p_0, g_0)) < \varepsilon.$$

The function $f(p, g)$, for fixed p, is called a **motion**, and the set

$$f(p, G) = \bigcup_{g \in G} f(p, g)$$

is called the **trajectory** of the point p.

A set $A \subseteq R$ is called **invariant** if

$$f(A, g) = \bigcup_{a \in A} f(a, g) = A$$

for every $g \in G$.

A set K of elements of the partially ordered group G will be called **relatively dense** if there exists an element $g_0 > e$ $(g_0 \in G)$ such that, whatever $g \in G$, there is an element $k \in K$ satisfying the inequalities $g < k < g g_0$.

A set $M \subseteq G$ will be called an ω**-set** if for every element $g \in G$ there exists an element $m \in M$ such that $g < m$.

Obviously, every relatively dense set of elements of the partially ordered group G is an ω-set.

Lemma 1. If the set $M \subseteq G$ is an ω-set (respectively, a relatively dense set), and $g_1, g_2 \in G$, then $g_1 M g_2$ is also an ω-set (respectively, a relatively dense set).

Proof. Let $g_1, g_2 \in G$, $M \subseteq G$, and let M be an ω-set. Then for any $h \in G$ there exists an element $m \in M$ such that $g_1^{-1} h g_2^{-1} < m$. Hence $h < g_1 m g_2 \in g_1 M g_2$, and, consequently, $g_1 M g_2$ is an ω-set.

Suppose now that $g_1, g_2 \in G$, $M \subseteq G$, and M is relatively dense. Then there exists $\tilde{g} \in G$ such that $\tilde{g} > e$, and for any $h \in G$ there is an element $m \in M$ satisfying the inequalities

$$g_1^{-1} h g_2^{-1} < m < g_1^{-1} h g_2^{-1} \tilde{g}.$$

In this case

$$h < g_1 m g_2 < h g_2^{-1} \tilde{g} g_2.$$

Since $g_2^{-1} \tilde{g} g_2 > e$, the set $g_1 M g_2$ is relatively dense. The lemma is proved.

Definition 1. An element $g_0 \in G$ will be called a **period** of the point $p \in R$ if $f(p, g_0) = p$.

Denote by G_p the set of all periods of the point p. This set forms a group.

Indeed, let $g_1, g_2 \in G_p$. Then

$$f(p, g_1 g_2) = f(f(p, g_1), g_2) = f(p, g_2) = p$$

and $f(p, g_2^{-1}) = p$, and therefore $g_1 g_2 \in G_p$ and $g_2^{-1} \in G_p$.

Lemma 2. If $p \in R$, and $g \in G$, then

$$G_{f(p,g)} = g^{-1} G_{pg}.$$

Proof. Let $p \in R$, $g \in G$, and $g_0 \in G_p$. Then

$$f(p, g g^{-1} g_0 g) = f(p, g),$$

and the element $g^{-1} g_0 g$ is a period of the point $f(p, g)$, i.e.

$$g^{-1} g_0 g \in G_{f(p,g)}.$$

Now let $g_1 \in G_{f(p,g)}$, i.e. $f(p, g g_1) = f(p, g)$. Then $f(p, g g_1 g^{-1}) = p$. Hence $g g_1 g^{-1} \in G_p$. Therefore $g_1 \in g^{-1} G_{pg}$. The lemma is proved.

Denote by $N(G_p)$ the normalizer of the set G_p in the group G [6].

Lemma 3. In order that $G_{f(p,g)} = G_p$ for some element $g \in G$, it is necessary and sufficient that $g \in N(G_p)$.

Proof. Let $p \in R$, and $g \in N(G_p)$. Then $g^{-1}G_{pg} = G_p$, and, according to Lemma 2, the group $G_{f(p,g)} = G_p$.

If, however, $G_{f(p,g)} = G_p$ for some $g \in G$, then by the same lemma $g^{-1}G_{pg} = G_p$ and $g \in N(G_p)$.

Corollary 1. The equality $G_{f(p,g)} = G_p$ holds for all $g \in G$ if and only if $N(G_p) = G$, i.e. G_p is a normal divisor of the group G.

Definition 2. A point $p \in R$ and a motion $f(p,g)$ will be called **weakly special** (**weakly cyclic, weakly periodic**) if the group of periods of the point p does not coincide with the identity group (is an ω-set, a relatively dense set).

Denote by A_1, A_2, A_3, respectively, the sets of all weakly special, weakly cyclic, and weakly periodic points.

Definition 3. A point $p \in R$ and a motion $f(p,g)$ will be called **special** (**cyclic, periodic**) if they are weakly special (weakly cyclic, weakly periodic) and

$$G_{f(p,g)} = G_p$$

for every $g \in G$.

Denote by B_1, B_2, B_3, respectively, the sets of all special, cyclic, and periodic points.

The following holds.

Theorem 1. The sets A_i, B_i $(i = 1, 2, 3)$ are invariant.

Proof. The invariance of B_i follows directly from Definition 3.

It follows from Lemma 1 that if G_p is an ω-set (respectively, a dense set), and $g \in G$, then $g^{-1}G_p^*g$ is also an ω-set (respectively, a dense set). Hence, on the basis of Lemma 2 we conclude that the sets A_i $(i = 1, 2, 3)$ are invariant.

According to Definitions 2 and 3, the following inclusions hold:

$$A_1 \supset A_2 \supset A_3, \qquad B_1 \supset B_2 \supset B_3, \qquad B_i \subset A_i \quad (i = 1, 2, 3).$$

Moreover,

$$B_2 = B_1 \cap A_2, \qquad B_3 = B_1 \cap A_3.$$

Thus, among the sets A_i, B_i $(i = 1, 2, 3)$ defined above, there are intersections. Furthermore, all of them are contained in A_1. At the same time, for example, the set of special motions does not coincide with the set of weakly special motions. Indeed, by Corollary 1, a weakly special motion $f(p,g)$ is special if and only if G_p is a normal divisor of the group G. In this connection, it is of interest to construct a partition of the set A_1, induced by the subsets A_2, A_3, B_1, B_2, B_3 [4, 7], i.e. a partition containing the minimal number of classes and such that each of the six sets $A_1, A_2, A_3, B_1, B_2, B_3$ defined above is the union of certain classes of this partition.

Taking into account the inclusions that hold between the sets A_i, B_i, $(i = 1, 2, 3)$, it is not hard to observe that the indicated partition may consist of the following six classes of motions: S_1—the set of all periodic motions; S_2—the set of all weakly periodic motions that are not special; S_3—the set of all cyclic motions that are not periodic; S_4—the set of all weakly cyclic motions that are neither cyclic nor weakly periodic; S_5—the set of all special motions that are not cyclic; S_6—the set of all weakly special motions that are neither special nor weakly cyclic.

It remains to show that each of these six classes is nonempty.

Example 1. Consider the set H of all continuous, strictly increasing functions on the entire real line. Introduce an operation in H as follows: if $h_1, h_2 \in H$, then $h_1 h_2 = h_2(h_1(x))$. The resulting function is continuous and strictly increasing. The operation introduced is associative, and the set H forms a group. The identity of the group is the function $e(x) \equiv x$. The inverse element for $h \in H$ is the inverse function $h^{-1}(x)$, which is continuous and strictly increasing.

In the group H introduce a partial order as follows: we shall say that $h_1 > h_2$ if the value of the function $h_1(x) > h_2(x)$ for all real x. The axioms of partial order are verified directly.

Define the dynamical system $[R_1, H, f_1]$, where R_1 is the line, H is the partially ordered group constructed, and f_1 is a function such that

$$f_1(x,h) = h(x)$$

for all $x \in R_1,\ h \in H$.

For the dynamical system constructed, the motion $f_1(x,h)$ is weakly special for any $x \in R_1$. The group of all periods of the point x is

$$H_x = \{h,\ h \in H : h(x) = x\}.$$

The set H_x is not an ω-set, since for the function $h(x) \equiv x + d,\ d > 0$, there does not exist a single function $h_0 \in H_x$ such that $h_0 > h$. Consequently, the motion $f_1(x,h)$ is not weakly cyclic.

Suppose now that $h_1 \in H_x$ is a period of the point $f_1(x,h)$ for any $h \in H$, i.e. if $h \in H$, then $f_1(x, hh_1) = f_1(x,h)$ or $h_1(h(x)) =$

$= h(x)$. But this equality holds for all $h \in H$ if and only if $h_1(x) = e(x)$.

Thus, the motion $f_1(x, h)$ is weakly special and at the same time is neither weakly cyclic nor special; consequently, the class S_6 is nonempty.

Example 2. Let W be the additive group of all continuous functions on the line, and let $w_1, w_2 \in W$. We shall assume that $w_1 > w_2$ if $w_1(x) > w_2(x)$ for all x.

As the space R take the line R_1, and define the dynamical system $[R_1, W, f_2]$ as follows: $f_2(x, w) = x + w(0)$ for any $x \in R_1$, $w \in W$. The axioms of a dynamical system are verified directly.

For any $x \in R_1$ the motion $f_2(x, w)$ is special. The group of periods of the point x is the set of functions passing through the origin. It is not an ω-set, since for the function $w(x) = d > 0$ there is no period w_0 of the point x greater than w. Thus, the motion $f_2(x, w)$ is special and is not cyclic; consequently, the class S_5 is nonempty.

Example 3. Let U be the group of all complex numbers with respect to addition. If $u_1, u_2 \in U$ and $u_1 = \alpha_1 + i\beta_1,\ u_2 = \alpha_2 + i\beta_2$, then we shall assume that $u_1 > u_2$ if $\alpha_1 > \alpha_2,\ \beta_1 > \beta_2$ simultaneously.

Define the dynamical system $[R_2, U, f_3]$, where R_2 is the plane, U is the group just indicated, and f_3 is the function which assigns to each point $(a, b) \in R_2$ and element $u = \alpha + i\beta$ the point

$$f_3((a, b), u) = (a + \alpha - \beta,\ b + \alpha - \beta).$$

The axioms of a dynamical system are verified directly.

The group of periods of any point is the bisector of the first and third quadrants of the plane U. Obviously, it is an ω-set, and therefore the motion $f_3((a, b), u)$ is cyclic. However, it is not difficult to see that the group of periods of the point (a, b) is not a relatively dense set. Consequently, the motion $f_3((a, b), u)$ is not periodic. This shows that the class S_3 is nonempty.

Example 4. Let V be the group of all square matrices of order two with positive determinant and ordinary multiplication as the group operation. If $v_1, v_2 \in V$, then we shall assume that $v_1 > v_2$ if $\det v_1 > \det v_2$.

To each point (a, b) of the plane R_2 assign the matrix (ab). Define the dynamical system $[R_2, V, f_4]$ as follows: for any point $(a, b) \in R_2$ and any

$$v = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

put

$$f_4((a, b), v) = (ab) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (aa_{11} + ba_{21},\ aa_{12} + ba_{22}).$$

The fulfillment of the axioms of a dynamical system is easily verified.

It can be shown that in this example all motions are weakly special. Moreover, for a point (a, b) with nonzero coordinates the periods have the form

$$\begin{pmatrix} \dfrac{a-bc}{a} & d \\ c & \dfrac{b-ad}{b} \end{pmatrix},$$

where c and d are chosen so that the determinant of the matrix is positive. The motion $f_4((a, b), v)$ is not special. Indeed, consider the point

$$(ab)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = (a, 2b)$$

on the trajectory of the point (a, b). Then

$$f_4\left((a, 2b), \begin{pmatrix} \dfrac{a-bc}{a} & d \\ c & \dfrac{b-ad}{b} \end{pmatrix}\right) = (a + bc,\, 2b - ad).$$

Hence it is clear that, among all periods of the point (a, b), the only matrix which is a period for the point $(a, 2b)$ will be the identity matrix.

We shall now show that the set of periods of the point (a, b) is relatively dense.

Let a matrix C have $\det C > 1$, and let D be an arbitrary matrix of the group V. Consider the interval $D < v < DC$. Let γ be such a real number that

$$\det D < \gamma < \det(DC).$$

There exists such a period v of the point (a, b) that $\det v = \gamma$. To see this, it is enough to note that the equation

$$\frac{a-bc}{a}\,\frac{b-ad}{b} - cd = \gamma$$

is solvable with respect to c and d. Then $D < v < DC$. Consequently, the set of periods of the point (a, b) is relatively dense and the motion $f_4((a, b), v)$ is weakly periodic.

Example 4 shows that the class S_2 is nonempty.

Example 5. Consider the direct product $G = U \times V$, where U and V are defined in Examples 3 and 4. Introduce in G a partial order as follows: if $g, h \in G$, then we shall say that $g > h$ if the first component of the element g is greater than the first component of the element h in the sense of the partial order in the group U.

Let $R = R_2 \times R_2$. Define the dynamical system $[R, G, f_5]$ as follows: for any pair $p = (p_1, p_2) \in R$ and $g = (u, v) \in G$, put $f_5(p, g) = (q_1, q_2)$, where $q_1, q_2 \in R_2$ and $q_1 = f_3(p_1, u)$, $q_2 = f_4(p_2, v)$.

The motion $f_5(p, g)$ is weakly cyclic. Indeed, the group of all periods of the point p is the set $G_p = U_{p_1} \times V_{p_2}$. Since U_{p_1} is an ω-set in U, according to the order introduced in the group G, G_p is an ω-set in G.

The motion $f_5(p, g)$ is not special, since the motion $f_4(p_2, v)$ is not special. Finally, the motion $f_5(p, g)$ is not weakly periodic, since U_{p_1} is not relatively dense in U, and consequently the set G_p is not relatively dense in the group G.

Example 5 establishes that the class S_4 is nonempty.

It is obvious that the class S_1 is also nonempty. To verify this, it is enough to note that S_1 includes periodic motions in ordinary dynamical systems.

Thus, the following holds.

Theorem 2. *The partition of the set of all weakly special motions, induced by the sets* A_2, A_3, B_1, B_2, B_3*, consists of the classes* S_i $(i = 1, 2, 3, 4, 5, 6)$*. In this case:*

$$A_j = \bigcup_{i=1}^{8-2j} S_i, \qquad B_j = \bigcup_{i=1}^{4-j} S_{2i-1} \quad (j = 1, 2, 3).$$

For ordinary dynamical systems all the sets A_j, B_j $(j = 1, 2, 3)$ coincide and contain only periodic motions. It follows from Theorem 2 that for partially ordered dynamical systems all these sets of motions are distinct.

Let us also note that the class S_1 of periodic motions includes all **stationary** motions, i.e., motions $f(p, g)$ for which $G_p = G$. This follows from the fact that the group G is always relatively dense in G. In connection with this, the class S_1 of periodic motions can be divided into two classes: S_1', of stationary motions, and S_1'', of periodic motions that are not stationary.

It is not difficult to establish that in the case of commutative groups

$$A_i = B_i, \quad S_{2i} = \Lambda, \quad S_{2i-1} \neq \Lambda \quad (i = 1, 2, 3).$$

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