ON LEBESGUE INTEGRATION IN CONSTRUCTIVE ANALYSIS

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Abstract

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MATHEMATICS

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ON LEBESGUE INTEGRATION IN CONSTRUCTIVE ANALYSIS

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This note is devoted to problems of Lebesgue integration on the segment $0\triangle 1$ of constructive functions of a real variable (c.f.r.v.), defined everywhere on $0\triangle 1$. It uses the definitions and results of the papers ($^{1-9}$).

In what follows the letters a, b serve as variables for rational numbers, x as a variable for real duplexes (^{4}).

An exact disjunctive segment covering Φ of the segment $0\triangle 1$ (^{8}) will be called simply a **covering**, if $\vartheta_{\mathrm{l}}(\Phi_0) = 0$ and $\vartheta_{\mathrm{r}}(\Phi_1) = 1$. Everywhere, a defined c.f.r.v. f will be called simply a **function**, if

$$\forall x\,((x \leqslant 0 \supset f(x) = f(0))\ \&\ (x \geqslant 1 \supset f(x) = f(1))).$$

It is clear that, in studying integration problems for the indicated class of c.f.r.v., it is sufficient to restrict oneself to considering functions.

For a function f and a covering Φ, denote by f/Φ the function such that

$$\forall kx \left(x \in \Phi_k \supset (f/\Phi)(x) = f(\vartheta_{\mathrm{l}}(\Phi_k)) + \frac{f(\vartheta_{\mathrm{r}}(\Phi_k)) - f(\vartheta_{\mathrm{l}}(\Phi_k))}{|\Phi_k|}(x - \vartheta_{\mathrm{l}}(\Phi_k)) \right).$$

By **pseudopolygonal functions** we shall mean functions which are pseudopolygonal in the sense of (^{5}). By **systems of constructive objects of the given type** we shall mean systems of words (^{1}) of this type, and by **sequences of functions** (coverings) normal algorithms $\mathfrak{A}$ such that, for every natural n, $\mathfrak{A}_n$ is a function (covering).

Definition 1. Let α be a letter not belonging to the alphabet $\{0,\ |,\ -,\ /,\ \diamond\}$. Words of the form

$$\alpha a_0 \alpha a_1 \alpha \dots \alpha a_n \alpha b_0 \alpha b_1 \alpha \dots \alpha b_n \alpha,$$

where n is a natural number, a_i and b_i are given rational numbers $(0 \leqslant i \leqslant n)$, $0 = a_0 < a_1 < \dots < a_n = 1$, will be called **polygonal bases**.

One can construct normal algorithms $\mathfrak{K}$ and $\mathfrak{I}$ such that, for every polygonal basis F, where

$$F = \alpha a_0 \alpha a_1 \alpha \dots \alpha a_n \alpha b_0 \alpha b_1 \alpha \dots \alpha b_n \alpha,$$

$\mathfrak{K}_F$ is a function and

$$\forall i x \left(1 \leqslant i \leqslant n \;\&\; a_{i-1} \leqslant x \leqslant a_i \supset \mathfrak{K}_F(x) = a_{i-1} + \frac{b_i - b_{i-1}}{a_i - a_{i-1}} (x - a_{i-1}) \right);$$

$\mathfrak{I}_F$ is an everywhere defined and everywhere differentiable c.f.r.v. for which

$$\forall x \left((\mathfrak{I}_F)'(x) = \mathfrak{K}_F(x) \right).$$

For polygonal skeletons one can define the operations of addition (denote it by $\dot{+}$), subtraction ($\dot{-}$), and absolute value ($|\;|_\circ$) so that, for any polygonal skeletons F and G and any real duplex x, one has

$$\widetilde{\mathfrak{R}}_{(F \dot{+} G)}(x) = \widetilde{\mathfrak{R}}_F(x) + \widetilde{\mathfrak{R}}_G(x), \qquad \widetilde{\mathfrak{R}}_{(F \dot{-} G)}(x) = \widetilde{\mathfrak{R}}_F(x) - \widetilde{\mathfrak{R}}_G(x),$$

$$\widetilde{\mathfrak{R}}_{|F|_\circ}(x) = \left| \widetilde{\mathfrak{R}}_F(x) \right|.$$

Definition 2. Let f be a function. We shall say that f is **Lebesgue integrable on the segment** $0 \triangle 1$ and write $f \in (\mathcal{L})$, if there exists a sequence of polygonal skeletons $\{F_n\}_n$ such that

$$\forall x m \, \exists k \, \forall n \left(n \geq k \supset \left| \widetilde{\mathfrak{R}}_{F_n}(x) - f(x) \right| < \frac{1}{m+1} \right), \tag{1}$$

$$\forall m \, \exists l \, \forall n k \left(n \geq l \supset \tilde{\mathfrak{I}}_{F_n \dot{-} F_{n+k}|_\circ}(1) - \tilde{\mathfrak{I}}_{F_n \dot{-} F_{n+k}|_\circ}(0) < \frac{1}{m+1} \right).$$

It is clear that for functions f, g such that $f \in (\mathcal{L})$, $g \in (\mathcal{L})$, and for a real duplex y, one has $|f| \in (\mathcal{L})$, $(f+g) \in (\mathcal{L})$, $(y \cdot f) \in (\mathcal{L})$.

Theorem 1. Let f be a function of weakly bounded variation on $0 \triangle 1$ (^{8}). Then f is Riemann integrable on $0 \triangle 1$ (for the definition see (^{9})), $f \in (\mathcal{L})$.

Definition 3. For a function f, $f \in (\mathcal{L})$, and real duplexes v, y, z, where $0 \leq y \leq 1$ and $0 \leq z \leq 1$, we shall say that v is the **value of the Lebesgue integral of f with limits** y, z, and write

$$v = \int_y^z f\,dx,$$

if there exists a c.f.d.s. $\mathfrak{F}$, defined everywhere on $0\triangle 1$ and differentiable on $0\nabla 1$, for which $\forall x(0 < x < 1 \supset \mathfrak{F}'(x) = f(x))$, $v = \mathfrak{F}(z) - \mathfrak{F}(y)$.

We shall also use the customary abbreviated notation, which we explain by an example. Let f be a function, $f \in (\mathcal{L})$; let y, z be real duplexes, $0 \le y \le 1$, $0 \le z \le 1$; let $P(v)$ be a predicate. Suppose that no occurrence of v in P lies in the scope of a quantifier binding v, y, z. Then we write

$$P\left(\int_y^z f\,dx\right) \Longleftrightarrow \forall v\left(v = \int_y^z f\,dx \supset P(v)\right).$$

Theorem 2. Let f be a function, $f \in (\mathcal{L})$. Then there exists an everywhere-defined and everywhere differentiable c.f.d.s. $\mathfrak{F}$ such that $\forall x(\mathfrak{F}'(x) = f(x))$, $\mathfrak{F}$ is a c.f.d.s. of bounded variation on $0\triangle 1$, and

$$\bigvee_0^1 \mathfrak{F} = \int_0^1 |f|\,dx.$$

If $\{F_n\}_n$ is a sequence of polygonal skeletons such that (1) is satisfied, then

$$\forall m\,\exists k\,\forall nx\left(n \ge k \supset \left|\tilde{\mathfrak{I}}_{F_n}(x) - \mathfrak{F}(x)\right| \le \frac{1}{m+1}(|x|+1)\right).$$

Theorem 3. Let f be a function. Then $f \in (\mathcal{L})$ if and only if there exist a c.f.d.s. $\mathfrak{F}$, defined everywhere on $0\triangle 1$ and differentiable on $0\nabla 1$, and a sequence of polygonal skeletons $\{T_n\}_n$ such that $\forall x(0 < x < 1 \supset \mathfrak{F}'(x) = f(x))$, and for every system of rational numbers $\{c_i\}_{i=0}^p$, for which $0 \le c_0 \le c_1 \le \dots \le c_p \le 1$, and every natural n,

$$\sum_{i=1}^p \left|\left(\mathfrak{F}(c_i) - \widetilde{\mathfrak{R}}_{T_n}(c_i)\right) - \left(\mathfrak{F}(c_{i-1}) - \widetilde{\mathfrak{R}}_{T_n}(c_{i-1})\right)\right| < \frac{1}{n+1}.$$

Theorem 4. Let $\{f_n\}_n$ be a sequence of functions, and let f be a function, such that $\forall n\,(f_n \in (\mathcal{L}))$, $\forall xm\,\exists k\,\forall n\left(n \ge k \supset |f_n(x) - f(x)| < \dfrac{1}{m+1}\right)$, and

$$\forall m\,\exists l\,\forall nk\left(n \ge l \supset \int_0^1 |f_n - f_{n+k}|\,dx < \frac{1}{m+1}\right).$$

Then $f \in (\mathcal{L})$ and

$$\forall m\,\exists l\,\forall n \left(n \geq l \supset \int_0^1 |f_n - f|\,dx < \frac{1}{m+1} \right).$$

Theorem 5. Let f be a function, $f \in (\mathcal{L})$, and let n be a natural number. Then there exists a pseudopolygonal function g such that $g \in (\mathcal{L})$ and

$$\forall x \left(|g(x) - f(x)| < \frac{1}{n+1} \right).$$

Theorem 6. Let f be a pseudopolygonal function. Then $f \in (\mathcal{L})$ if and only if there exists a sequence of coverings $\{ {}^n\Phi \}_n$ such that, for every natural n, $f/{}^n\Phi$ is a uniformly continuous function,

$$\forall k \left(\sum_{i=0}^{k} |\, {}^n\Phi_i \,| < \frac{1}{n+1} \right),$$

$\forall k\,\exists l\,({}^{n+1}\Phi_k \subseteq {}^n\Phi_l)$, and

$$\forall m\,\exists l\,\forall nk \left(n \geq l \supset \int_0^1 |\, f/{}^n\Phi - f/{}^{n+k}\Phi \,|\,dx < \frac{1}{m+1} \right).$$

Theorem 7. Let f, g be functions, $f \in (\mathcal{L})$, $\forall ab\,(|g(a) - g(b)| \leq |f(a) - f(b)|)$. Then $g \in (\mathcal{L})$.

Theorem 8. Let f, g be functions and let $\{g_n\}_n$ be a sequence of functions such that $f \in (\mathcal{L})$,

$$\forall nab\,(|g_n(a) - g_n(b)| \leq |f(a) - f(b)|)\,,$$

$$\forall xm\,\exists k\,\forall n \left(n \geq k \supset |g_n(x) - g(x)| < \frac{1}{m+1} \right).$$

Then $g \in (\mathcal{L})$, $\forall n\,(g_n \in (\mathcal{L}))$, and

$$\forall m\,\exists k\,\forall n \left(n \geq k \supset \int_0^1 |g_n - g|\,dx < \frac{1}{m+1} \right).$$

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Note: Figure translations are in progress. See original paper for figures.

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