EVOLUTIONARY GRAPHS AND THEIR APPLICATION TO THE INVESTIGATION AND SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

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Abstract

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EVOLUTIONARY GRAPHS AND THEIR APPLICATION TO THE INVESTIGATION AND SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

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We shall call an evolutionary graph a set of black and white vertices connected by arcs, constructed according to the following three rules:

1. An arc of the graph, $\widehat{kj}$, connecting vertex k with vertex j in the direction from k to j, indicated by the arrow, transforms the element X_k of the linear space H, given at the beginning of the arc, into the element $X_j \in H$ at its end according to the equation $X_j = \mathcal{L}_{jk} X_k$, where $\mathcal{L}_{jk}$ is the linear operator of the arc (Fig. 1a).
2. At black vertices there is only one outgoing arc and any number $M \geq 1$ of incoming arcs. $X_1, X_2, \dots, X_k, \dots, X_{M_j}$, defined at the ends of the incoming arcs, are transformed by the black vertex into $X_j = \sum_{1,\dots,M_j} X_k$, defined on the outgoing arc and assigned to the vertex itself (Fig. 1b).
3. At white vertices there is only one incoming arc and any number $N \geq 1$ of outgoing arcs. X_j at the end of the incoming arc of the white vertex j, assigned to the vertex itself, is transformed into $X_k = X_j,\ k = 1, 2, \dots, N_j$, defined at the beginnings of the outgoing arcs (Fig. 1c).

The external vertices of the graph have free arcs not connected with other vertices of the graph. If all free arcs of an external vertex are incoming (outgoing), then it is called an input (output) vertex. In an ordinary graph X is given only on the free input arcs. If X is given on at least one outgoing arc, then the graph is called loaded.

Between evolutionary graphs and systems of linear equations there exists a correspondence determined by Theorems 1 and 2.

Theorem 1. To every evolutionary graph there corresponds a system of equations "read off" from it, consisting of the equations of the black vertices $1, 2, 3, \dots, j, \dots, l, \dots, n$:

$$Y_j = \sum_k^{(j)} \mathcal{L}_{jk} Y_k + \sum_{k'}^{(j)} \mathcal{L}_{jk'} X_{k'}, \qquad j = 1, 2, \dots, n, \tag{1}$$

where $Y_k (X_{k'})$ are elements of H in the black (white) vertices connected with the given vertex j by arcs with operators $\mathcal{L}_{jk}$ ($\mathcal{L}_{jk'}$), and from the equations of the white vertices

$$X_{k'} = \mathcal{L}_{k'l} Y_l \quad \text{or} \quad X_{k'} = \mathcal{L}_{k'l'} X_{l'}, \qquad k' = 1, 2, \dots, h', \tag{2}$$

where $Y_l (X_{l'})$ are elements of H in the unique black (white) vertex connected with the given k' by arcs with operators $\mathcal{L}_{k'l} (\mathcal{L}_{k'l'})$. In constructing (1) and (2), the equations of rule 3, which reduce the number of X' s, are taken into account.

The proof of this theorem follows directly from rules 1, 2, and 3 defining an evolutionary graph.

To prove the converse theorem, consider the most general system of linear operational equations

$$\Phi_j(X_1, \dots, X_k, \dots, X_P,\ X_{P+1}, \dots, X_{P+q}, \dots, X_{P+Q}) = 0, \quad j = 1, 2, \dots, R, \tag{3}$$

where $\{X_k\}_1^P$ are the unknown, and $\{X_{P+q}\}_1^Q$ the prescribed elements of H; Φ_j are linear forms in them with operators $\mathcal{L}_{jk}$ as coefficients.

Let us divide (3) into three systems:

$$\begin{aligned} &1)\ \Phi_j(X_1, \dots, X_k, \dots, X_P) = Z_j'; \\ &2)\ \Phi_j(X_{P+1}, \dots, X_{P+Q}) = Z_j''; \\ &3)\ Z_j' = -Z_j''. \end{aligned} \tag{4}$$

Represent the first system by the graph shown on the left, and the second by the graph on the right in Fig. 2a. We shall call such graphs PR- and QR-graphs. Here the free arcs of the black vertices of the PR- and QR-graphs are directed toward one another, and in order to obtain from them the graph of system (3), one must invert the directions of the free arcs in the PR- or QR-graph according to the third system (4). Then we obtain the general $PR + QR$-graph representing (3).

From rules 1, 2, and 3 there follow the following operations of **arc inversion**:

A. When changing the direction of the arrow of the arc $\widehat{kj}$, one must replace $\mathcal{L}_{jk}$ by the inverse $\mathcal{L}_{kj} = \mathcal{L}_{jk}^{-1}$ (Fig. 1a). If $\mathcal{L}_{jk}^{-1}$ exists, then the arc $\widehat{kj}$ is called invertible.

B. When replacing an outgoing arc of a black vertex by one of the incoming arcs, one must change the signs of X on these arcs or on all the remaining ones (Fig. 1d); at white vertices the signs of X remain unchanged (Fig. 1e);

Fig. 1

Lemma. In order to perform N inversions of input arcs in a graph into output arcs, it is necessary and sufficient that the invertible input vertices be connected with N output vertices by N paths without common vertices. When N input arcs are inverted, N output arcs paired with them will be inverted simultaneously.

Hence it follows:

Theorem 2. Necessary and sufficient conditions for it to be possible to construct an ordinary evolutionary graph for system (3) are the following: the PR-graph must have $R''' + R''_+$ invertible arcs connecting black vertices pairwise with white ones, and the QR-graph $R' + R''_-$ invertible arcs, where R' are the vertices for which $Z''_j = 0$; R''' are the vertices for which $Z'_j = 0$; R'' are paired black vertices of the graphs;

$$R''_+ + R''_- = R''.$$

The total number of output vertices of the $PR+QR$-graph is equal to R, and the remaining $P+Q-R$ will be input vertices. If these conditions are not satisfied, one can always construct a loaded $PR+QR$-graph with the conditions $X = 0$ at certain black vertices.

Theorem 2 determines the minimal number of pairs of invertible vertices required for constructing a $PR+QR$-graph. In the general case, several variants of constructing a $PR+QR$-graph are possible for the selected input vertices in which the X_q are specified. Each variant of the $PR+QR$-graph is a program for solving system (3). By inverting arcs in it, one can obtain other $PR+QR$-graphs corresponding to a different partition of the variables X_k into unknown and specified ones. The problem of investigating and solving a system of equations by means of graphs reduces: 1) to constructing, for system (3), a programming $PR+QR$-graph in which X_{p+q} are specified on the input free arcs, while X_k are determined on the output arcs of the graph; 2) to simplifying the structure of this graph by means of transformation operations; 3) to "reading off" the system of equations from the simplified graph, which, according to (1) and (2), gives the solution of (3) in explicit form. The principal graph-transformation operation (the method of loaded paths), following from rules 1, 2, 3, makes it possible to find X_k at vertex k for a given X_q at vertex q with the aid of the operator $\mathcal{L}_{kq}$ in the equation $X_k = \mathcal{L}_{kq} X_q$, which replaces the arcs of the graph connecting q with k by a single arc with $\mathcal{L}_{kq}$. To do this: 1) all paths $(q,k)_t$, $t = 1, 2, \dots, \tau$, from q to k are found; 2) the path operators $\mathcal{L}[q,k]_t$ are computed;

Fig. 2

Fig. 2

Figure 1: Fig. 2

3) the path operators are summed, obtaining

$$\mathcal{L}_{kq} = \sum_{t=1}^{\tau} \mathcal{L}[q,k]_t.$$

If the path $[q,k]$ is simple, i.e., consists of identically oriented arcs $\widehat{12}, \widehat{23}, \widehat{34}, \widehat{45}, \dots, \widehat{jj+1}, \dots, N$ (Fig. 1) with operators $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_j, \dots, \mathcal{L}_{N-1}$, then

$$\mathcal{L}[q,k] = \mathcal{L}_{N-1} \dots \mathcal{L}_j \dots \mathcal{L}_2 \dots \mathcal{L}_1.$$

If the path $[q,k]$ contains feedback loops, as, for example, in Fig. 1 , then the feedback loop characterized by the loop operator

$$T_k^{(j)} = \mathcal{L}_{k''+n} \dots \mathcal{L}_{k''} \mathcal{L}_{k'-1} \dots \mathcal{L}_j$$

is replaced by an arc with the operator

$$\mathcal{L}_{\text{o.c}} = (E - T_k)^{-1},$$

inserted into the break between vertices $j'-1$ and j', where j' is the initial vertex of the loop (any vertex of the loop). As a result, for the graph of Fig. 1 we obtain

$$\mathcal{L}[q,k] = \mathcal{L}_{kj} = \mathcal{L}_k \dots \mathcal{L}_{k'} \dots \mathcal{L}_{j'} \mathcal{L}_{\text{o.c}} \mathcal{L}_{j'-1} \dots \mathcal{L}_j.$$

If there are many feedback loops, then the "loading" by them of the path between q and k is performed in the sequence of their arrangement along the path.

Example. A system of equations is given: 1) $X_1 - dX_3 = X_0$; 2) $X_3 - bX_1 = X_0$; 3) $aX_1 + cX_3 - X_5 = 0$, where X_0 is given, and X_1, X_3, and X_5 are unknown. Its $PR+QR$-graph is given in Fig. 2 . From the input vertex 0 to the output vertex 5 there are four paths: 02165, 0214365, 04365, and 0432165, loaded by the feedback loop 14321. In Fig. 2 a scheme is given of the evolution of the variables X along the path 02165; the other evolution schemes are analogous. Using-vesm

by the method of weighted paths, we obtain the solution of the system of equations in the form
$X_5 = \mathcal{L}_{50} X_0,\ \mathcal{L}_{50} = a(E-db)^{-1}(E+d) + c(E-bd)^{-1}(E+b).$

Definition. A path $[j,k]$ between vertices j and k is called **isolated from the vertex** l if the sum $\sum_{k',t}\mathcal{L}[l,t_{k'},k]=0$, where $\mathcal{L}[l,t_{k'},k]$ are the operators of weighted paths from l to k (Fig. 2e). Otherwise the path $[j,k]$ is not isolated from l. A contour (closed path) isolated and not isolated from the vertex l is defined analogously.

In contours S with $T_S=E$, isolated or compensated with respect to the input vertices (when several input vertices, each producing nonzero effects on the contour separately, by the selection of X_q give a zero effect in the sum), for some vertex of the contour j_S one may prescribe an arbitrary $X_j\in H$; the $X_{j'}$ at the remaining vertices of the contour are determined through X_j.

The X_k at the output vertices of an ordinary evolutionary graph are uniquely determined through the X_q of the input vertices of the graph and the X_j of contours with $T=E$, isolated or compensated with respect to the input vertices:

$$X_k=\sum_{q=1}^{Q}\sum_{t=1}^{\tau_{qk}}\mathcal{L}[q,k]_t X_q+\sum_{s=1}^{S}\sum_{t=1}^{\tau_{sk}}\mathcal{L}[j_s,k]_t X_{j_s}, \tag{5}$$

where $\mathcal{L}[j_s,k]$ are the operators of weighted paths from the indicated contours to k. The X_q in (5) are arbitrary if, by inverting arcs between the input vertices, it is impossible to construct paths isolated from the output vertices. In this case the X_q of the input vertices will be linearly independent.

Compatibility conditions for system (3). If for (3) one can construct an ordinary $PR+QR$-graph, then it is inconsistent only in exceptional cases, when the graph contains contours with $T=E$ that are not isolated (or not compensated) simultaneously from the input and output vertices. If for (3) one can construct only a weighted graph, then it is compatible only in exceptional cases, i.e., under a special choice of the operators $\mathcal{L}_{jk}$ and X_q, when the conditions $X=0$ at the black vertices are fulfilled. If contours with $T=E$ are absent in the $PR+QR$-graph, then, by the method of weighted paths, it is transformed into a QP-graph, and the solution (5) is represented in the form $X_k=\Phi_k(X_{P+1}\dots X_{P+q}\dots X_{P+Q})$. A necessary condition for the existence of nontrivial solutions of the homogeneous system (3) (there are no input vertices) is the existence in the $PR+QR$-graph of contours with $T=E$ connected with the output vertices.

Prohibition rule. From rules 1, 2, 3 it follows that, in evolutionary graphs, vertices with several input and output arcs do not exist; therefore, in transformations of graphs the merging of white vertices with black ones is forbidden. Otherwise, an unambiguous correspondence cannot be established between graphs and systems of equations (3), as, for example, in the case of Mason signal-flow graphs ([2,3]). The ambiguity of "reading off" a system of equations from a graph with vertices having several input and output arcs is illustrated by Fig. 1 , if in them the black and white vertices are merged into one.

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Note: Figure translations are in progress. See original paper for figures.